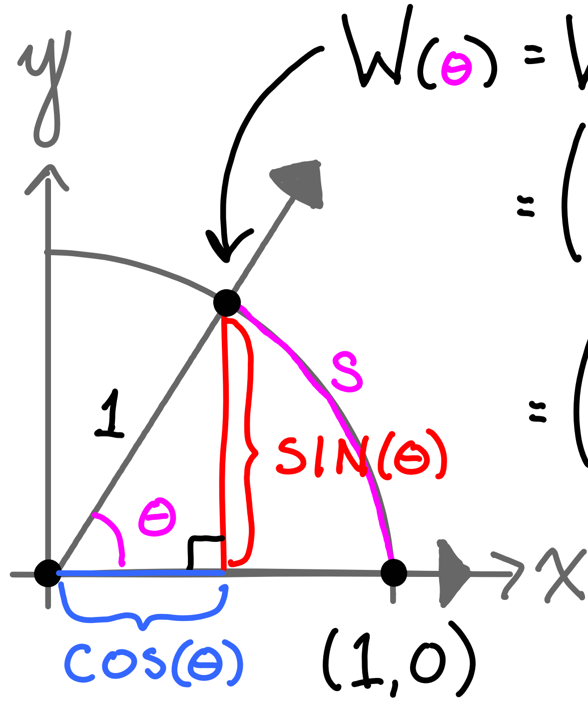
Sine, Cosine, and the Unit Circle

Recall the wrapping function

and $\theta := \frac{s}{r} \Leftrightarrow s = r\theta = \theta$ when $R=1$



$$W(\theta) = W(s) = (\cos(\theta), \sin(\theta))$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots,\ \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!},\ \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}\right)$$

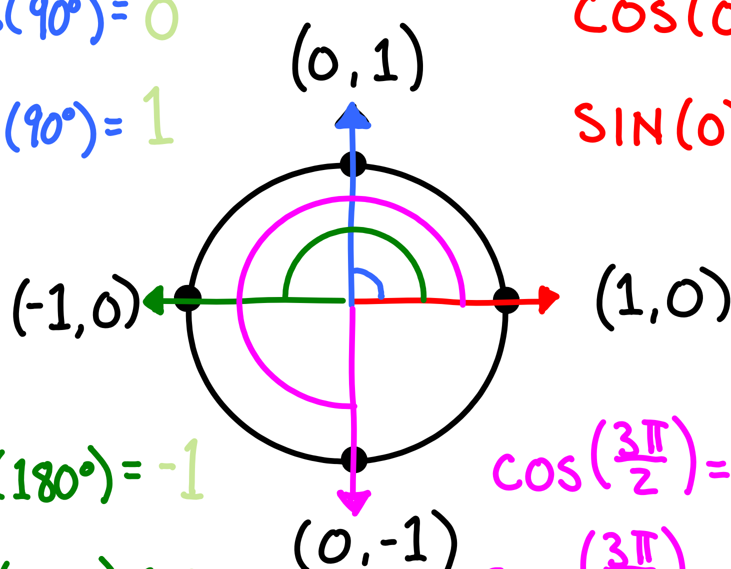
The Easy Angles

$\cos\left(\frac{\pi}{2}\right) = \cos(90°) = 0$

$\sin\left(\frac{\pi}{2}\right) = \sin(90°) = 1$

$\cos(0) = \cos(0°) = 1$

$\sin(0) = \sin(0°) = 0$



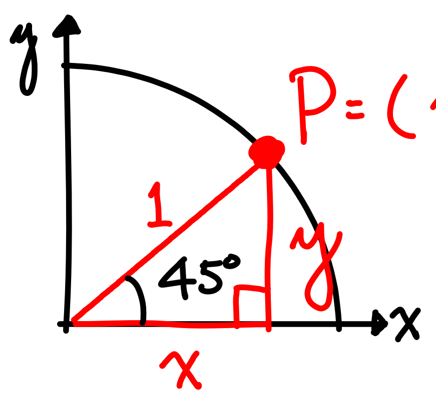
$\cos(\pi) = \cos(180°) = -1$

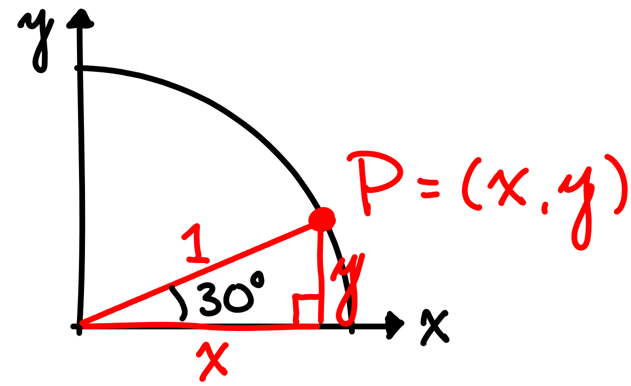
$\sin(\pi) = \sin(180°) = 0$

$\cos\left(\frac{3\pi}{2}\right) = \cos(270°) = 0$

$\sin\left(\frac{3\pi}{2}\right) = \sin(270°) = -1$

We want to know P for as many different θ's as possible. It turns out that we can find many by solving the following two triangles:





But first, some basic triangle facts

The Pythagorean Theorem

$$a^2 + b^2 = c^2$$

or

$$adj^2 + opp^2 = hyp^2$$

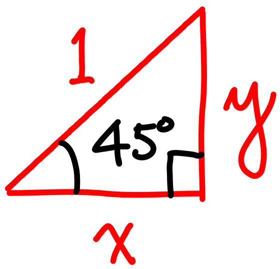
or

$$x^2 + y^2 = r^2$$

Triangle Fact: The sum of the angles of a triangle on a flat surface is $180°$

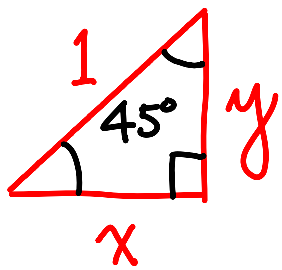
What if the surface isn't flat?

SOLVING THE 45° TRIANGLE



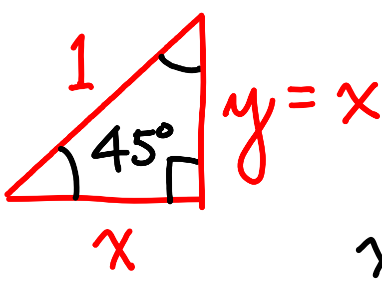
SINCE THE ANGLES ADD TO 180°

WE KNOW THE 3RD ANGLE IS ALSO 45°.



TWO UNKNOWNS. WE NEED TWO EQUATIONS

WHY?	THE PYTHAGOREAN THEOREM
$x = y$	$x^2 + y^2 = 1^2$

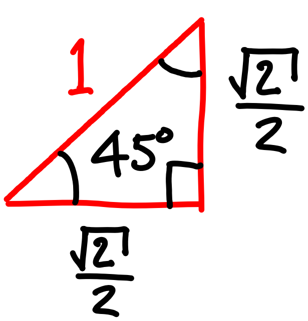


SOLVE:

$$x^2 + y^2 = 1^2 = 1$$

$$x = y \Rightarrow 1 = x^2 + x^2 = 2x^2$$

$$\Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2} = x = y}$$



$$\cos(45°) = \cos(\pi/4) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sin(45°) = \sin(\pi/4) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

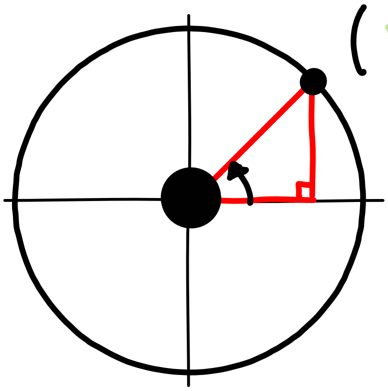
$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow$ $\boxed{\frac{\sqrt{2}}{2} \text{ AND } \frac{1}{\sqrt{2}} \text{ MAY BE USED INTERCHANGEABLY}}$

FRACTIONS <u>DO NOT</u> HAVE TO BE RATIONALIZED.

WHAT CAN YOU DO WITH THIS TRIANGLE?

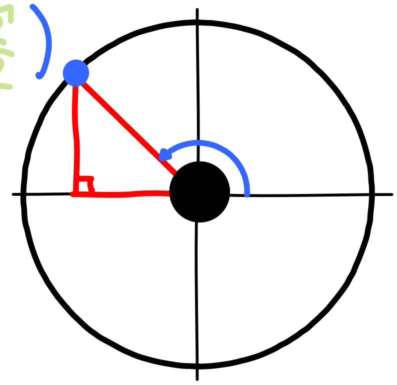
$\theta = 45° = \frac{\pi}{4}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

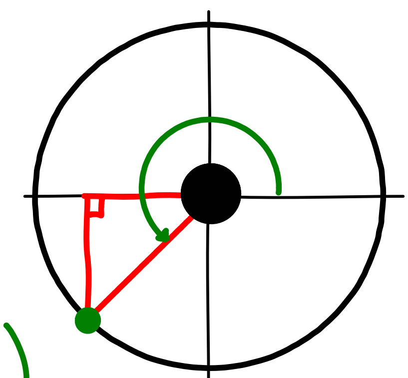


$\theta = 135° = \frac{3\pi}{4}$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

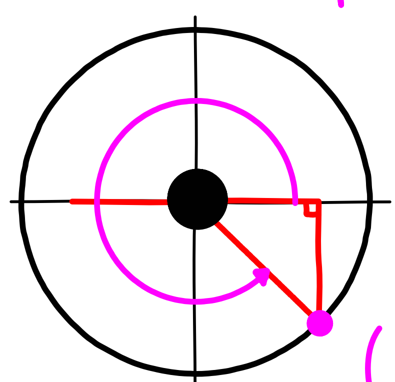


$\theta = 225° = \frac{5\pi}{4}$

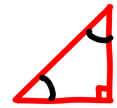


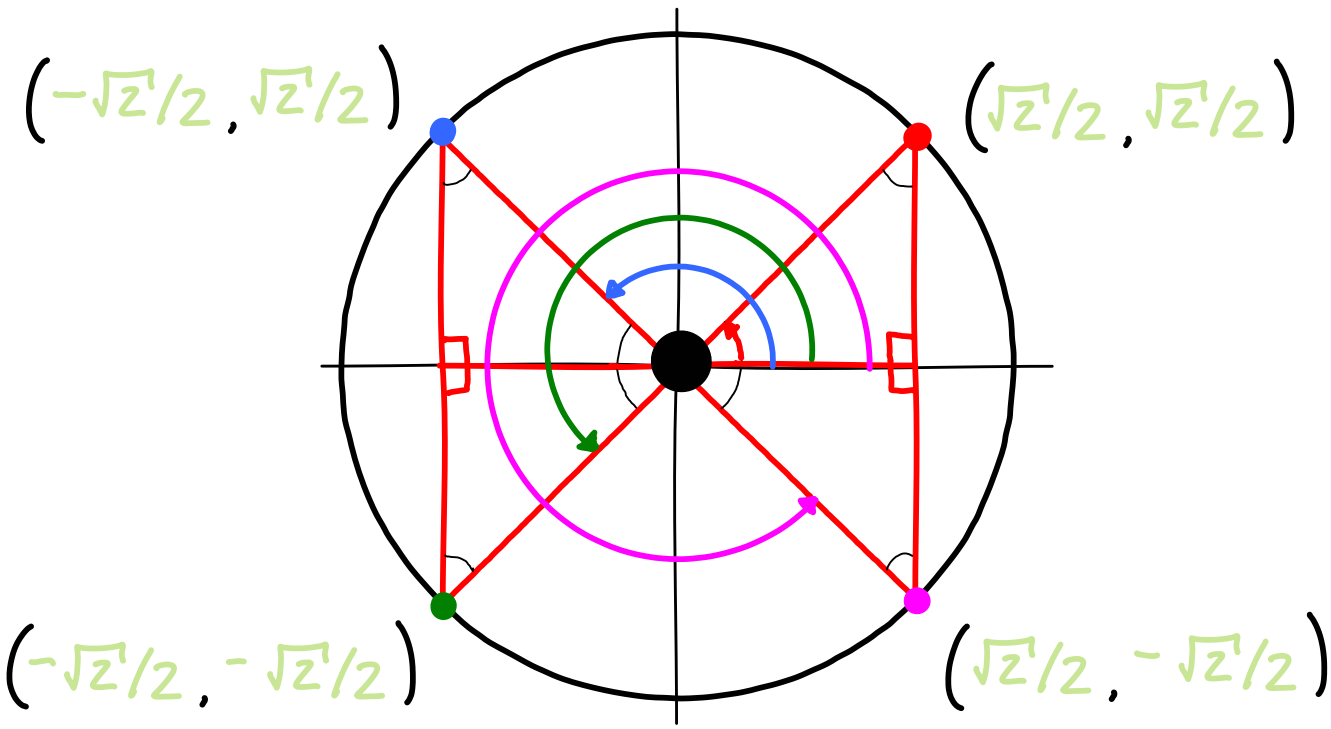
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$\theta = 315° = \frac{7\pi}{4}$



$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

WHAT YOU KNOW ABOUT THE UNIT CIRCLE BECAUSE OF THIS TRIANGLE 



$\sin(135°) = \sin(3\pi/4) = \sqrt{2}/2$

$\cos(135°) = \cos(3\pi/4) = -\sqrt{2}/2$

$\sin(45°) = \sin(\pi/4) = \sqrt{2}/2$

$\cos(45°) = \cos(\pi/4) = \sqrt{2}/2$

$\sin(225°) = \sin(5\pi/4) = -\sqrt{2}/2$

$\cos(225°) = \cos(5\pi/4) = -\sqrt{2}/2$

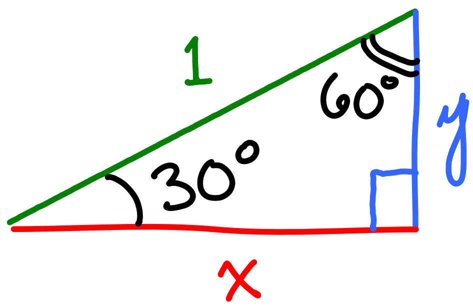
$\sin(315°) = \sin(7\pi/4) = -\sqrt{2}/2$

$\cos(315°) = \cos(7\pi/4) = \sqrt{2}/2$

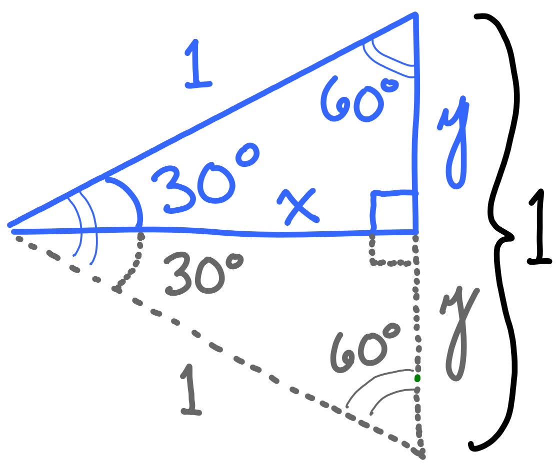
WHAT ABOUT $\cos(405°) = \cos(9\pi/4) =$

COSINE AND SINE ARE PERIODIC FUNCTIONS WITH PERIOD, $T = 2\pi$.

Solving the 30°/60° Triangle



The trick here is to solve a different triangle that is related to this one but easier to solve.



Put two of these triangles together to make an equilateral triangle

$$2y = 1 \Rightarrow y = 1/2$$

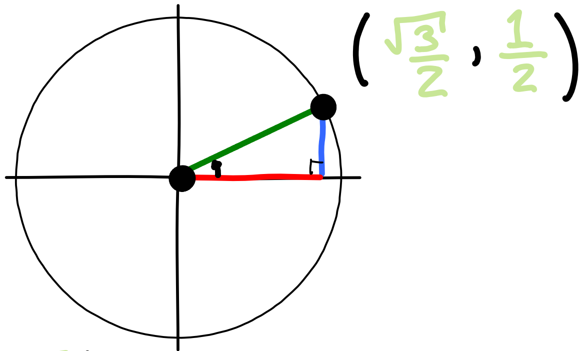
Now use the Pythagorean theorem

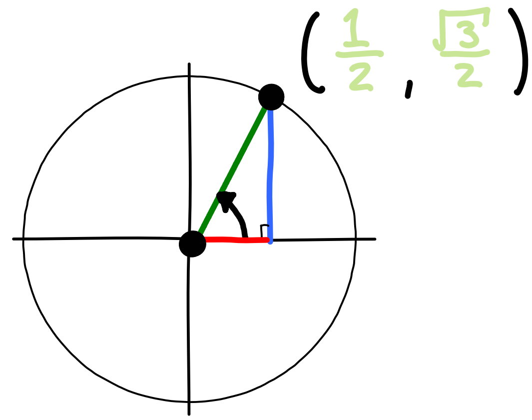
$$\Rightarrow \quad x^2 + y^2 = x^2 + (1/2)^2 = x^2 + 1/4 = r^2 = 1$$

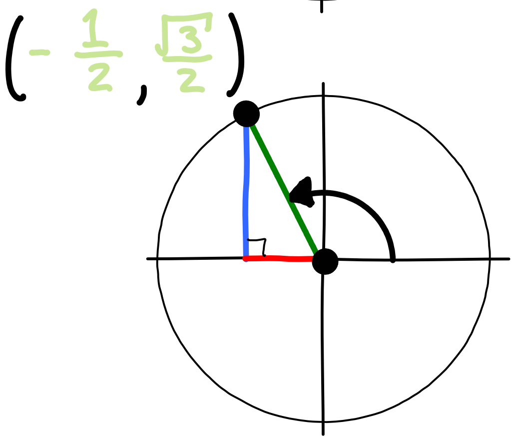
$$\Rightarrow \quad x^2 = 1 - 1/4 = 3/4$$

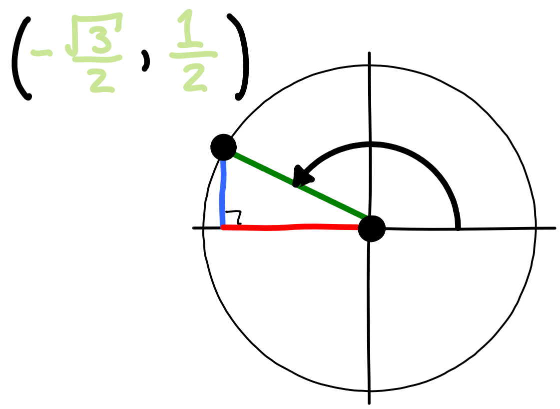
$$\Rightarrow \quad x = \sqrt{3/4} = \sqrt{3}/2$$

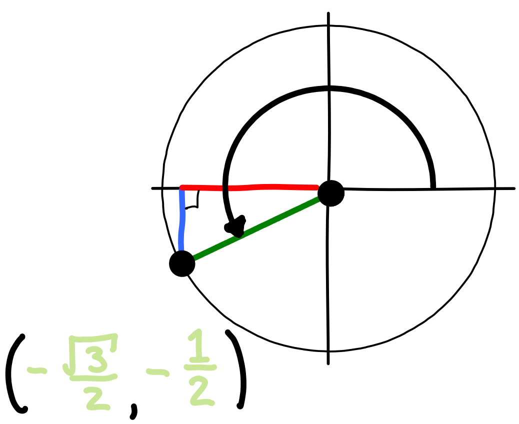
WHAT CAN YOU DO WITH THIS TRIANGLE?

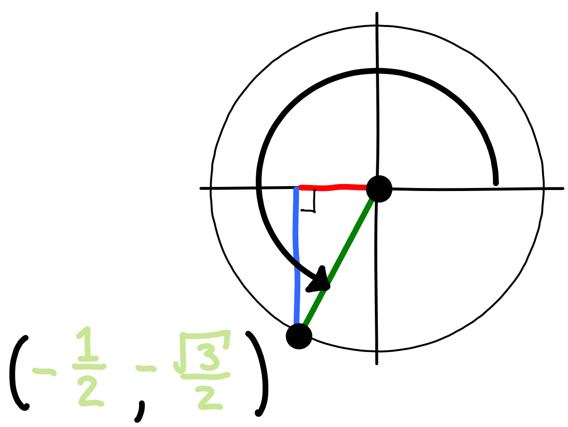


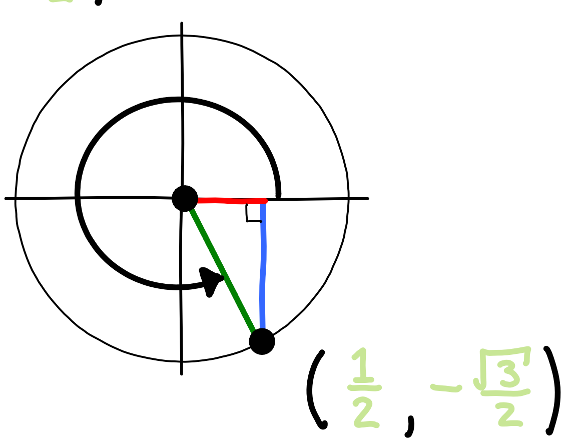


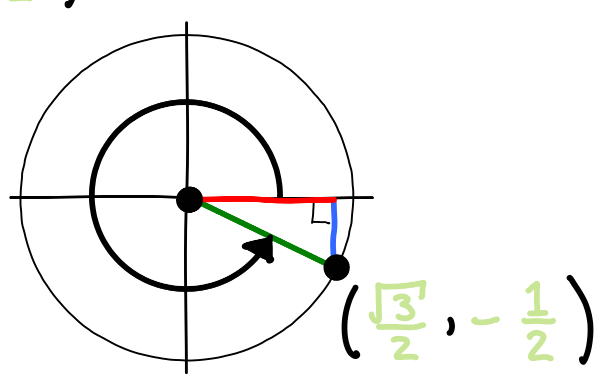


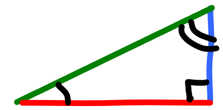


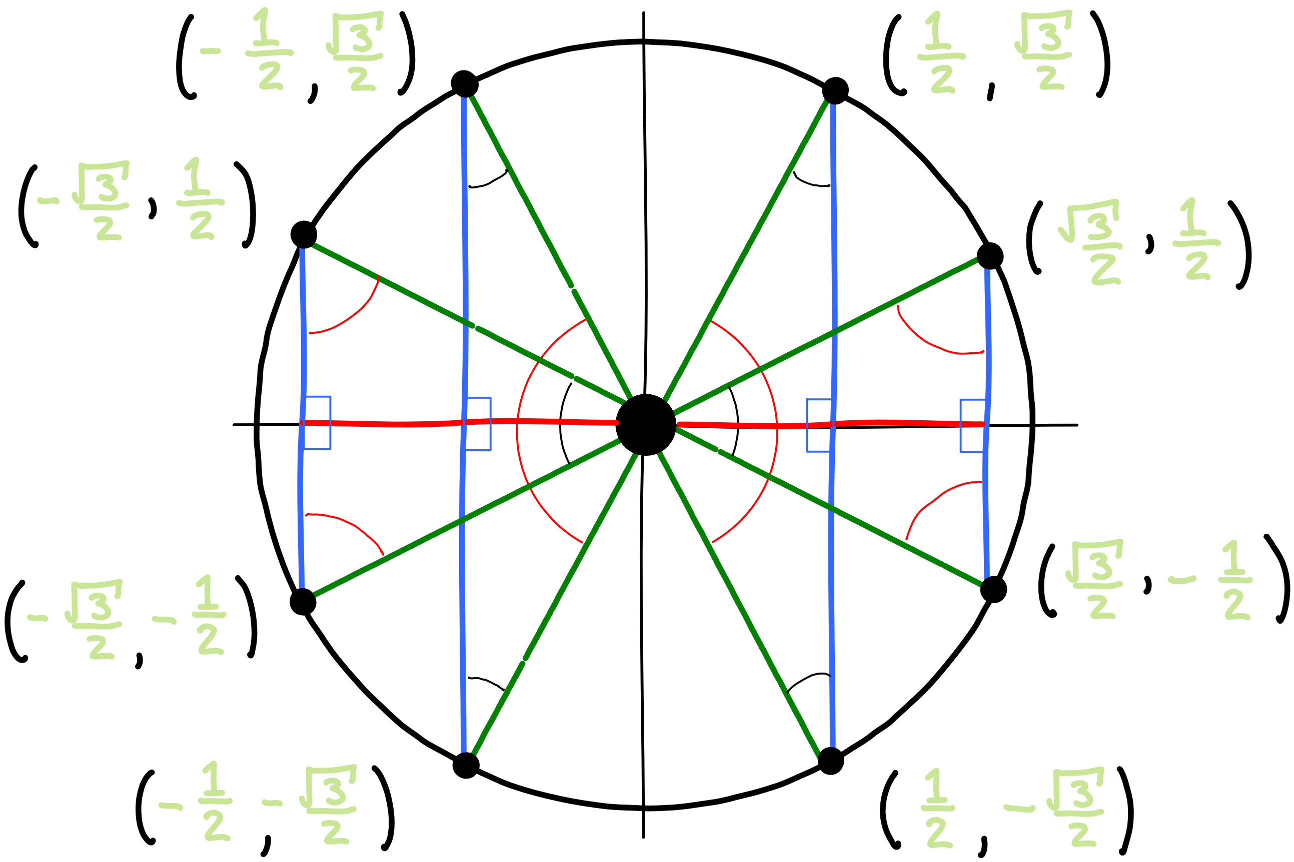








WHAT YOU KNOW ABOUT THE UNIT CIRCLE BECAUSE OF THIS TRIANGLE 



THE UNIT CIRCLE

A TOOL USED TO EVALUATE TRIGONOMETRIC FUNCTIONS.