Inverse Trigonometry

Doing Trigonometry Backwards

To solve for x in $3x^2+1=2$

we must undo what was done to x.

We must:

1) subtract (opposite of add)
2) divide (opposite of multiply)
3) take square root (opposite of squaring)

$f(x)=x-1$, $g(x)=\frac{x}{3}$, $h(x)=\sqrt{x}$

$3x^2+1=2$

$\Leftrightarrow f(3x^2+1)=f(2) \Leftrightarrow 3x^2+1-1=2-1$

$\Leftrightarrow 3x^2=1$

$\Leftrightarrow g(3x^2)=g(1) \Leftrightarrow 3x^2/3=1/3$

$\Leftrightarrow x^2=1/3$

$\Leftrightarrow h(x^2)=h(1/3) \Leftrightarrow \sqrt{x^2}=\sqrt{1/3}$

$$\Leftrightarrow |x|=1/\sqrt{3} \Rightarrow \begin{cases} x=1/\sqrt{3} \\ x=-1/\sqrt{3} \end{cases}$$

What is an Inverse Function?

If f takes x to y, then f^{-1} takes y back to x.

$$f: x \mapsto y \Rightarrow f^{-1}: y \mapsto x$$

If $f(x) = x+1$, then $f^{-1}(x) = x-1$

If $g(x) = x-1$, then $g^{-1}(x) = x+1$

$\Rightarrow$ f and g are inverses of one another.

$f^{-1}(f(x)) =$

$f(f^{-1}(x)) =$

$$\boxed{f(f^{-1}(x)) = x = f^{-1}(f(x)) \text{ and } (f^{-1})^{-1} = f}$$

Notation:

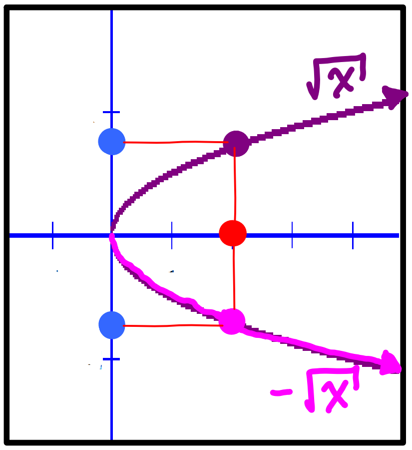
- x is a variable and $x^{-1} = 1/x$
- f is a function and $f^{-1} \neq 1/f$
- f^{-1} means "f inverse."

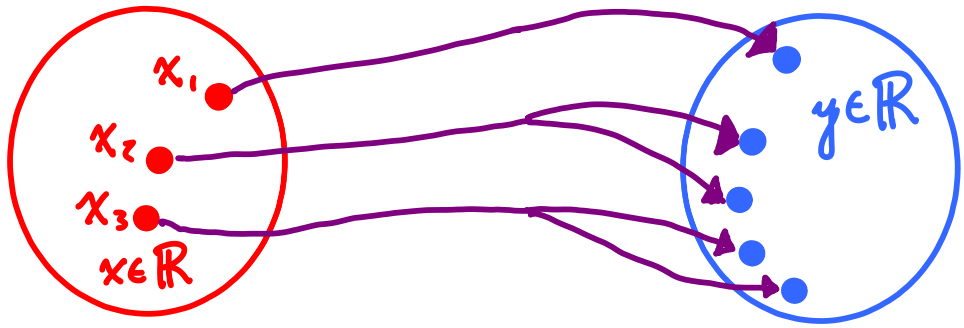
A function, by definition, cannot take one x to two different y's.

It's graph must pass the vertical line test.

Functions take one or more x's to one y

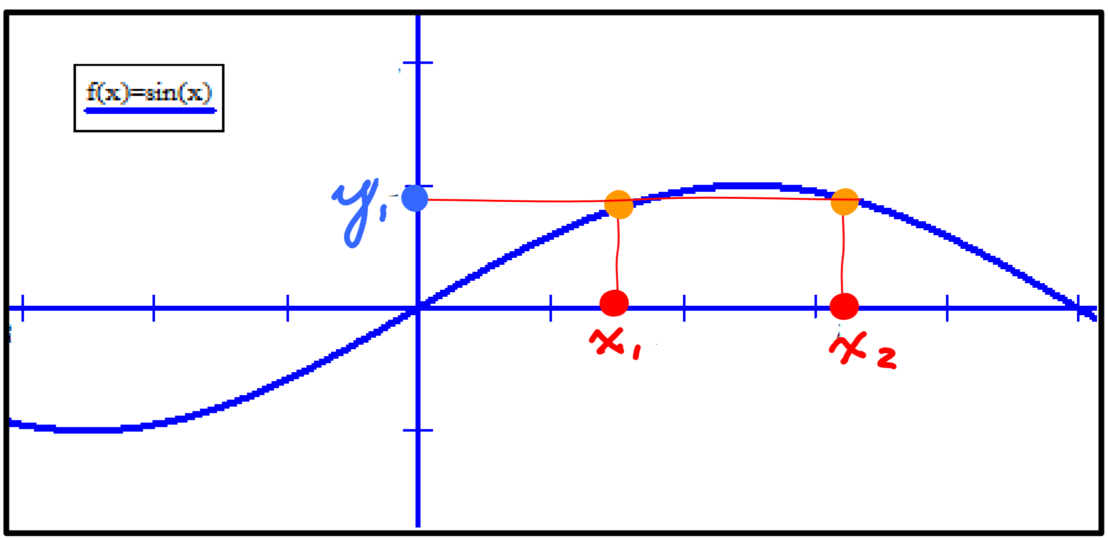
Functions can <u>NOT</u> take one x to two or more y's.

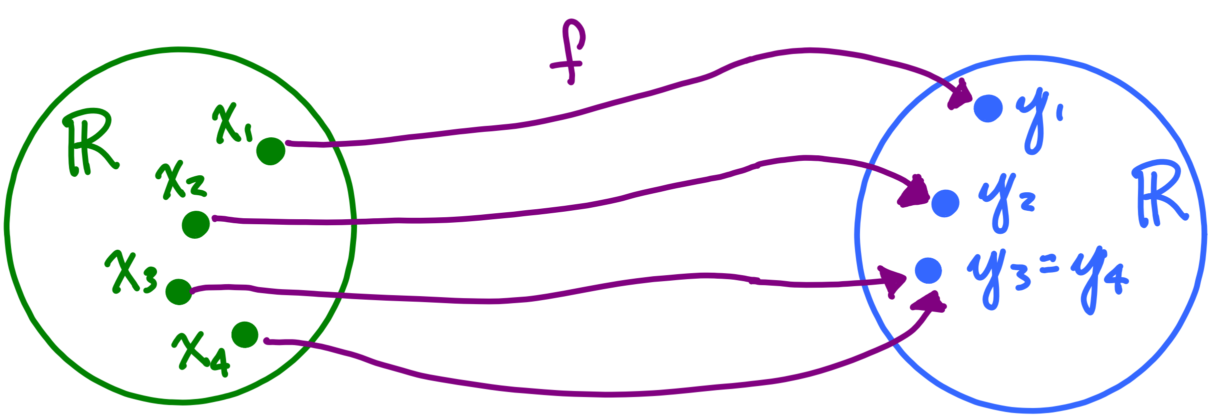




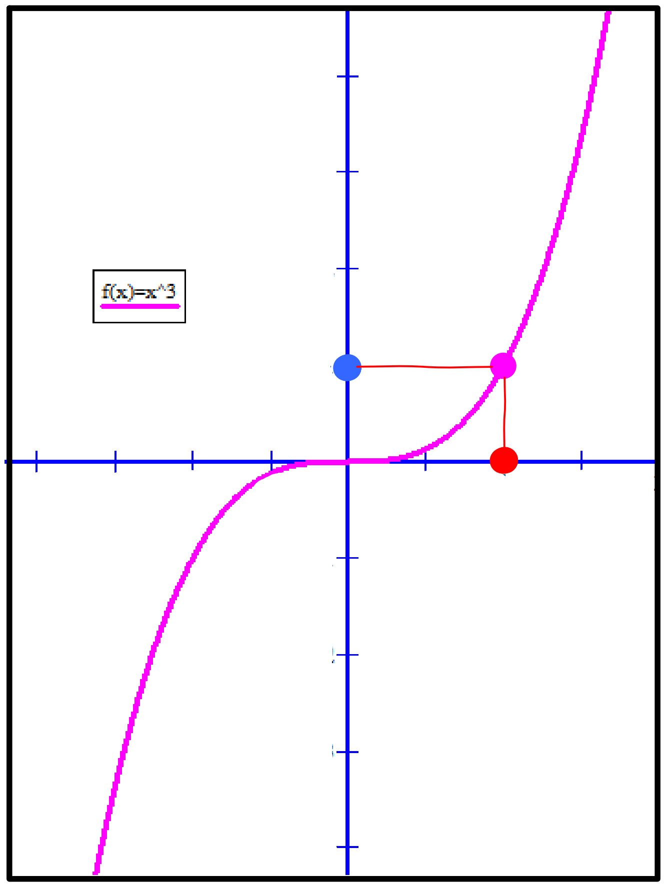
Not a Function

This is a Function

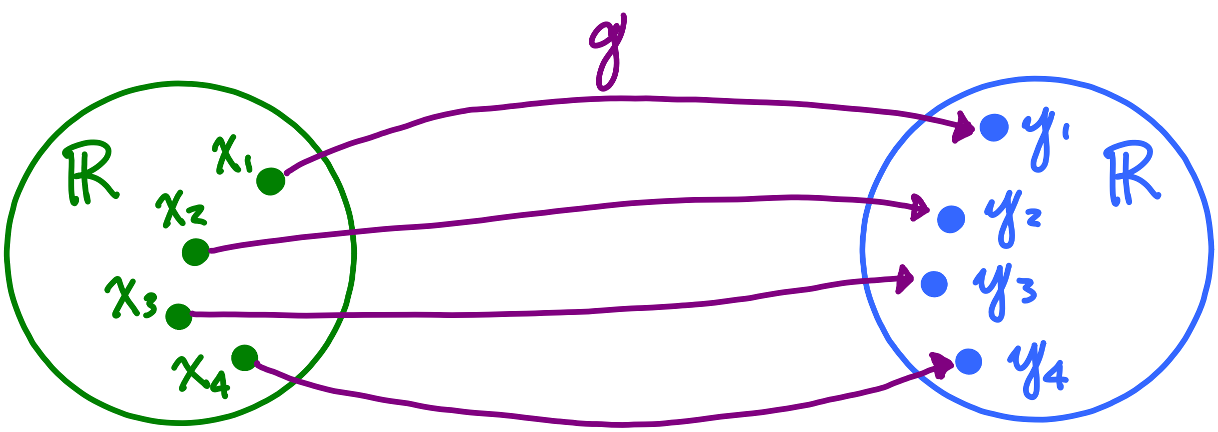




Some Functions are One-To-One

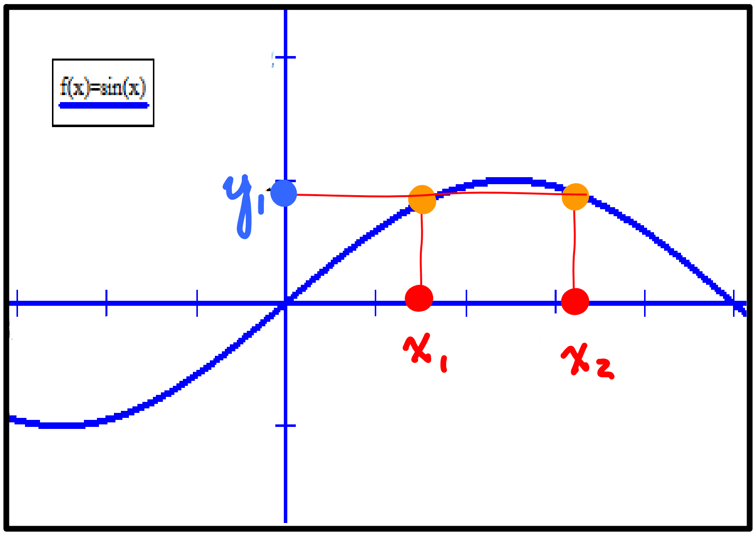


These One-To-One Functions Have Inverses



We know exactly which x each y came from.

TRIGONOMETRIC FUNCTIONS ARE NOT ONE-TO-ONE



DID y_1 COME FROM x_1 or x_2?

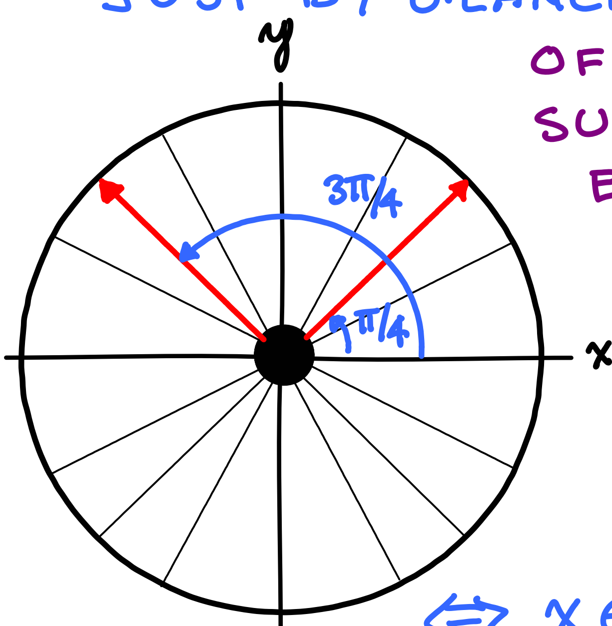
To solve for x in $\sin(x) = \frac{\sqrt{2}}{2}$

We must undo what sine did to x.

BUT THIS EXAMPLE IS TOO EASY.

WE KNOW THAT $\sin(\pi/4) = \sin(3\pi/4) = \frac{\sqrt{2}}{2}$

JUST BY GLANCING AT THE UNIT CIRCLE



OF COURSE ADDING OR SUBTRACTING 2π FROM EITHER OF THOSE ANGLES IS STILL A SOLUTION.

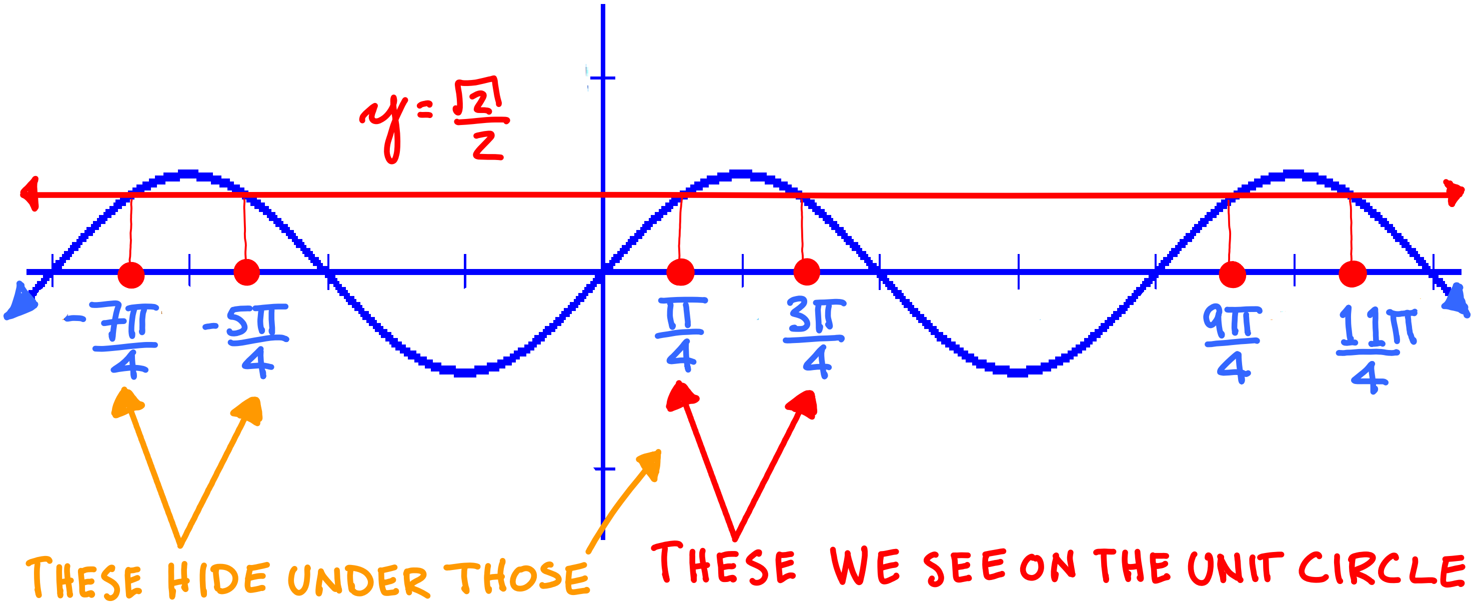
$$\sin(x) = \frac{\sqrt{2}}{2}$$

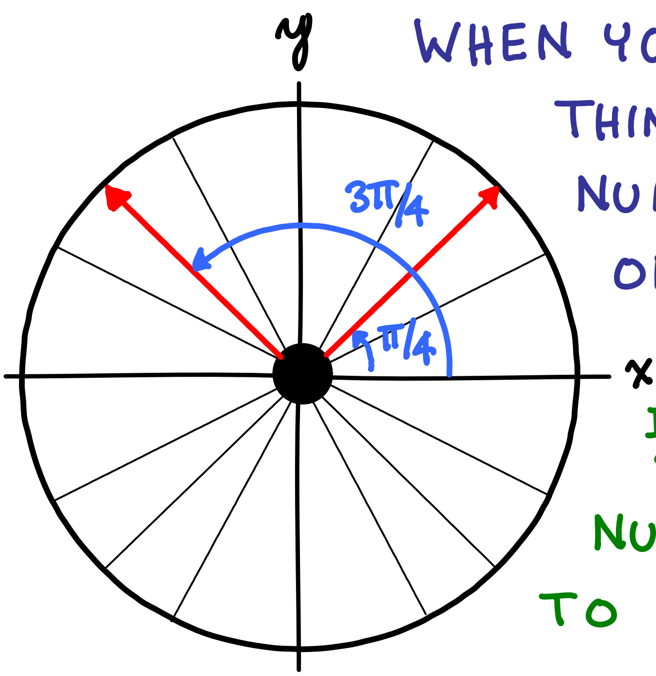
$$\Leftrightarrow x \in \{\pi/4 + 2n\pi,\ 3\pi/4 + 2n\pi;\ n \in \mathbb{Z}\}$$

$$= \left\{\ldots, -\frac{13\pi}{4}, -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots\right\}$$

$\sin(x) = \frac{\sqrt{2}}{2}$ HAS AN INFINITE NUMBER OF SOLUTIONS

YOU CAN SEE THEM ON THE GRAPH.





WHEN YOU THINK OF THE UNIT CIRCLE THINK OF IT AS AN INFINITE NUMBER OF CIRCLES STACKED ON TOP OF EACH OTHER.

UNDERNEATH THE ANGLES $\frac{\pi}{4}$ AND $\frac{3\pi}{4}$ ARE AN INFINITE NUMBER OF ANGLES EQUIVALENT TO $\frac{\pi}{4}$ AND $\frac{3\pi}{4}$

WHAT IF WE WANT TO SOLVE $\sin(x) = 0.79346$?

WE CAN'T SOLVE THIS BY LOOKING AT THE CIRCLE.

WE COULD ROUGHLY APPROXIMATE IT BY EYE.

BUT WE NEED TO DO BETTER THAN THAT

LET'S DESIGN A FUNCTION THAT GIVES US ONE POSSIBLE x, THAT A CERTAIN y CAME FROM.

THIS IS GOOD ENOUGH BECAUSE WE CAN FIGURE OUT THE REST OF THE x'S OURSELVES.

WE'LL CALL IT ARCSINE AND NICKNAME IT $\sin^{-1}$ OR SINE INVERSE.

WITH THIS FUNCTION, WE COULD SOLVE UGLY EQUATIONS. WE COULD APPLY IT TO BOTH SIDES OF THE EQUATION.

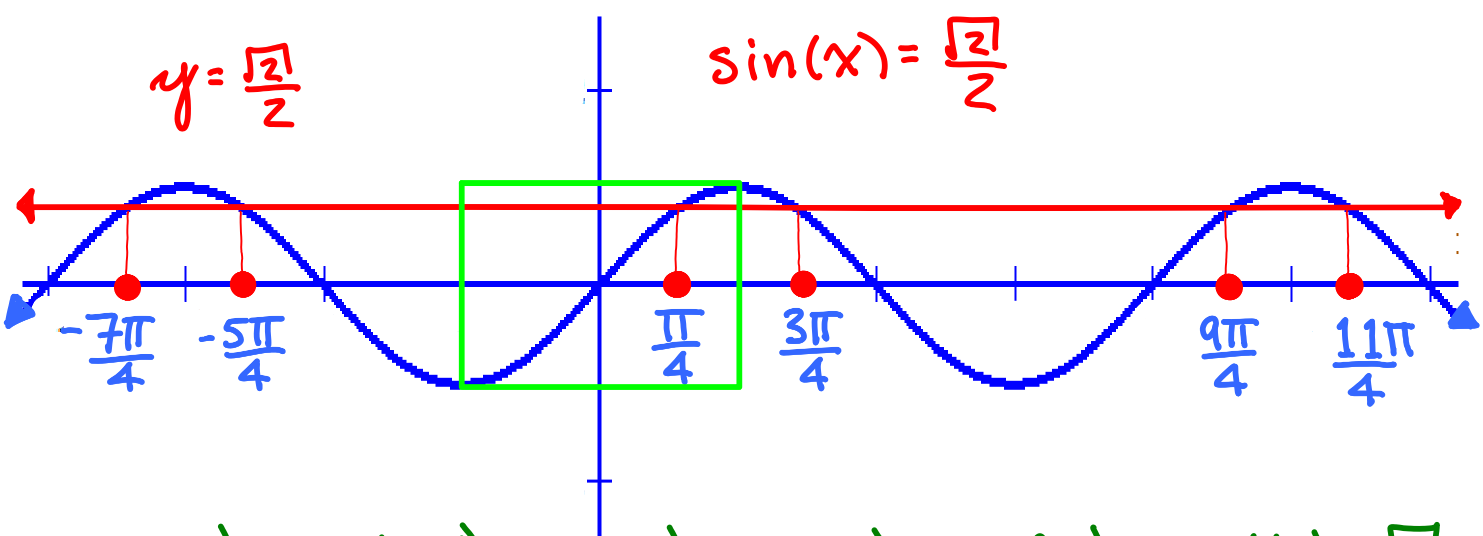
$$\sin(x) = 0.79346$$

$$\Rightarrow \sin^{-1}(\sin(x)) = \sin^{-1}(0.79346)$$

$$\Leftrightarrow x = \sin^{-1}(0.79346)$$ ← PLUG INTO CALCULATOR, IT GIVES US A NUMBER

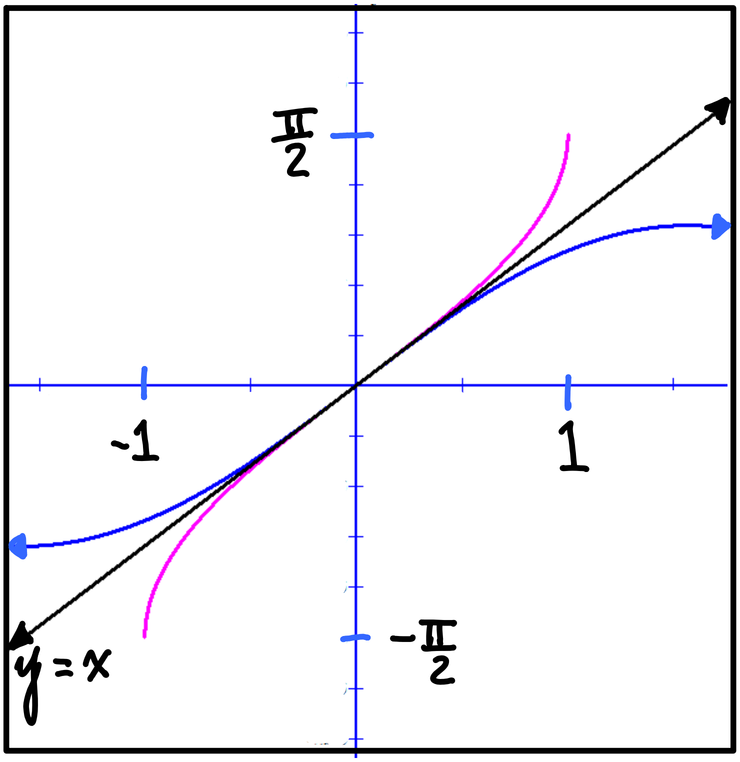
WE FIGURE OUT THE REST OURSELVES.

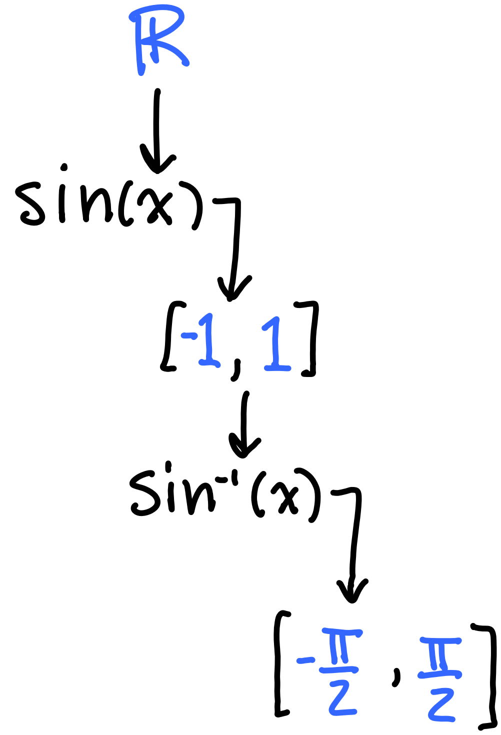
WE JUST NEED TO DECIDE WHICH x WE WANT.

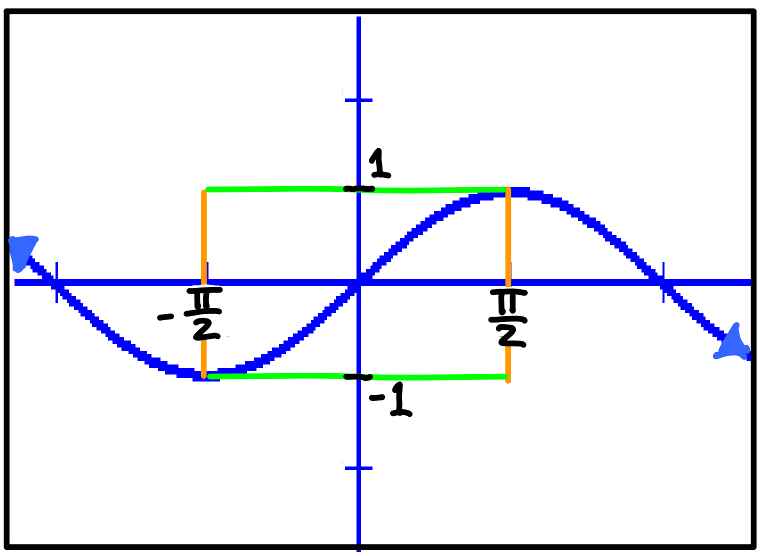


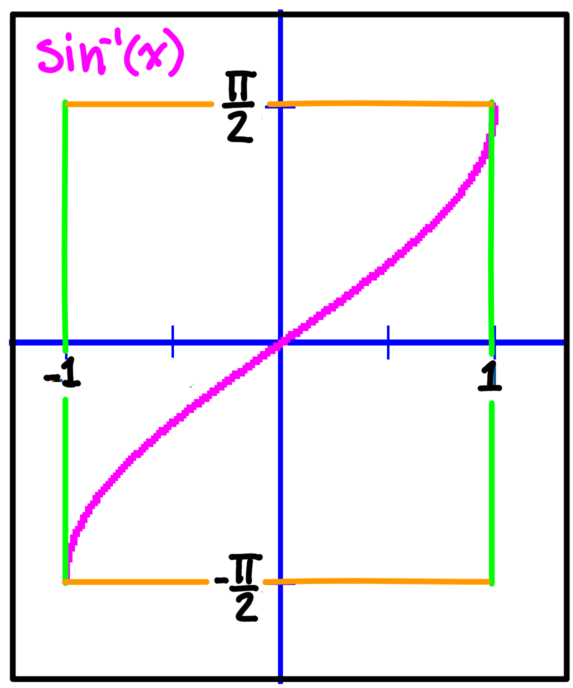
$$\sin\left(-\tfrac{7\pi}{4}\right) = \sin\left(-\tfrac{5\pi}{4}\right) = \sin\left(\tfrac{\pi}{4}\right) = \sin\left(\tfrac{3\pi}{4}\right) = \sin\left(\tfrac{9\pi}{4}\right) = \sin\left(\tfrac{11\pi}{4}\right) = \tfrac{\sqrt{2}}{2}$$

$\arcsin(x) = \operatorname{asin}(x) = \sin^{-1}(x)$







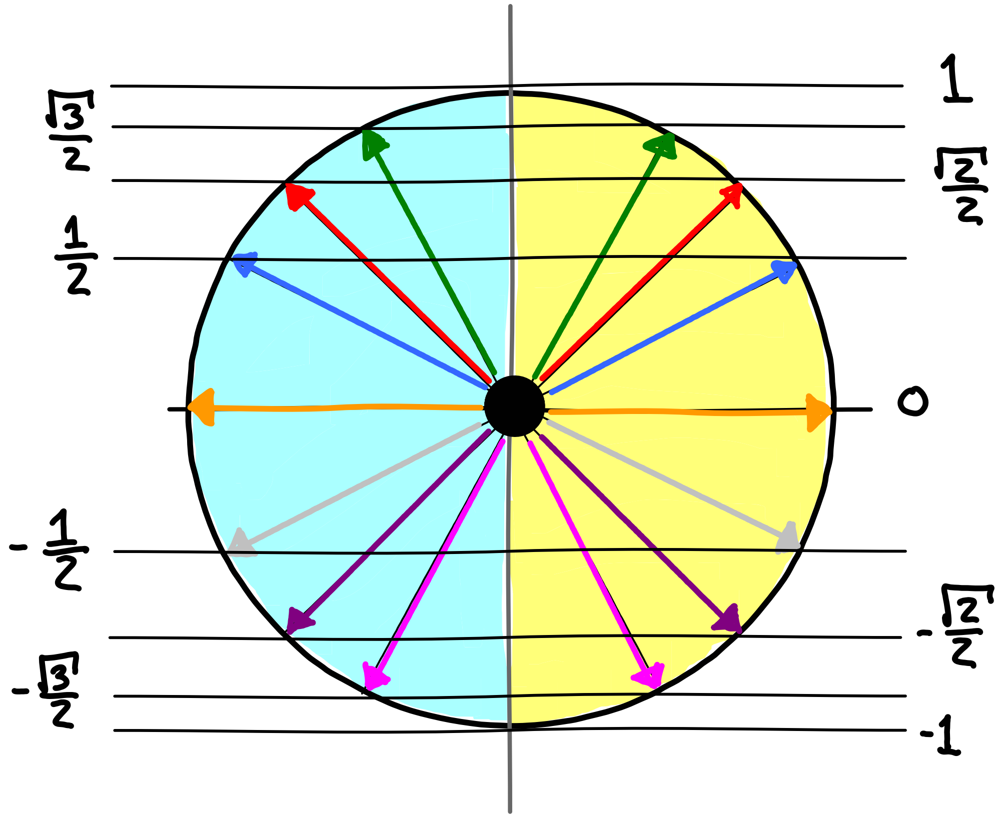


$\sin(x) \in [-1, 1]$

If $y \in [-1, 1]$ then $\sin^{-1}(y) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

otherwise $\sin^{-1}(y)$ is undefined.

SINE INVERSE WAS BORN TO CLEAN UP WHAT SINE LEAVES LYING AROUND.



THE LEFT HAND SIDE OF THE UNIT CIRCLE
IS A MIRROR IMAGE OF
THE RIGHT HAND SIDE OF THE UNIT CIRCLE
AS FAR AS SINE IS CONCERNED

WE CHOOSE THE RIGHT HAND SIDE

$\sin^{-1}(-1) = -\frac{\pi}{2}$	$\Rightarrow$	$\sin\left(-\frac{\pi}{2}\right) = -1$
$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$	$\Rightarrow$	$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$
$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$	$\Rightarrow$	$\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$	$\Rightarrow$	$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$
$\sin^{-1}(0) = 0$	$\Rightarrow$	$\sin(0) = 0$
$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$	$\Rightarrow$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$	$\Rightarrow$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$	$\Rightarrow$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$\sin^{-1}(1) = \frac{\pi}{2}$	$\Rightarrow$	$\sin\left(\frac{\pi}{2}\right) = 1$