Inverses and Arcsine

Doing Trigonometry Backwards

Inverse Review

To solve for x in $3x^2+1=2$

We must undo what was done to x.

1) Subtract (inverse of add)
2) Divide (inverse of multiply)
3) Square Root (inverse of Square)

$f(x)=x-1$ $\quad g(x)=\frac{x}{3}$ $\quad h(x)=\sqrt{x}$

$$3x^2+1=2$$

$$\Leftrightarrow f(3x^2+1)=f(2)$$

$$\Leftrightarrow g(3x^2)=g(1)$$

$$\Leftrightarrow h(x^2)=h(1/3)$$

$$x+1=0 \begin{cases} \Leftrightarrow x=-1 \\ \Leftarrow (x)^2=(-1)^2 \Leftrightarrow x^2=1 \begin{cases} \Rightarrow x=1 \\ \Rightarrow x=-1 \\ \Leftrightarrow x=\pm 1 \end{cases} \end{cases}$$

BE CAREFUL WHEN MANIPULATING EQUATIONS.

EACH NEW EQUATION IS DIFFERENT.

KNOW WHEN YOU ARE ADDING NEW SOLUTIONS,
LOSING SOLUTIONS,
AND WHEN YOUR NEW EQUATION HAS
THE SAME SOLUTIONS AS YOUR OLD ONE.

USE YOUR SYMBOLS CORRECTLY

$$\Rightarrow \qquad \Leftarrow \qquad \Leftrightarrow$$

What is an Inverse Function?

If f takes x to y, then f^{-1} takes y to x.

$$f: x \mapsto y \Rightarrow f^{-1}: y \mapsto x$$

If $f(x) = x+1$, then $f^{-1}(x) = x-1$

If $g(x) = x-1$, then $g^{-1}(x) = x+1$

$\Rightarrow$ f and g are inverses of one another.

$f^{-1}(f(x)) =$

$f(f^{-1}(x)) =$

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) \quad \text{and} \quad (f^{-1})^{-1} = f$$

Notation:

- x is a variable and $x^{-1} = 1/x$
- f is a function and $f^{-1} \neq 1/f$
- f^{-1} means "f inverse."

What operation does an exponent represent?

$$x^4 = x \cdot x \cdot x \cdot x$$

Positive exponents ⇒ Repetitive Multiplication

$$x^{-4} = \frac{1}{x^4} = \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}$$

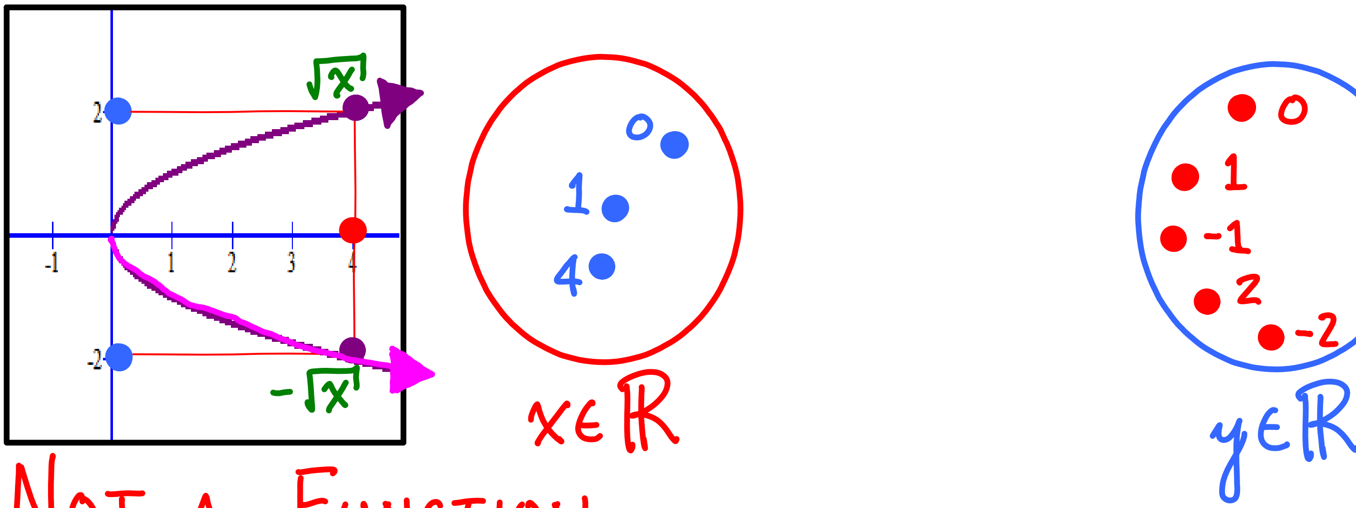
Negative exponents ⇒ Repetitive Division

The "inverse" of Multiplication is Division.

The -1 in f^{-1} is NOT an exponent

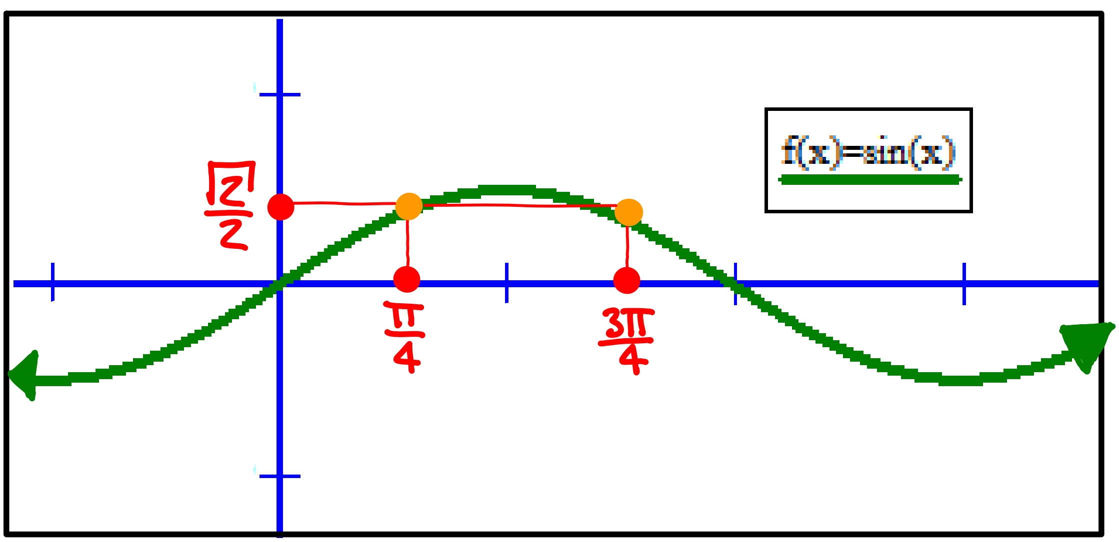
This -1 is referring to the inverse operation of f instead of the inverse of multiplication that is represented by negative exponents

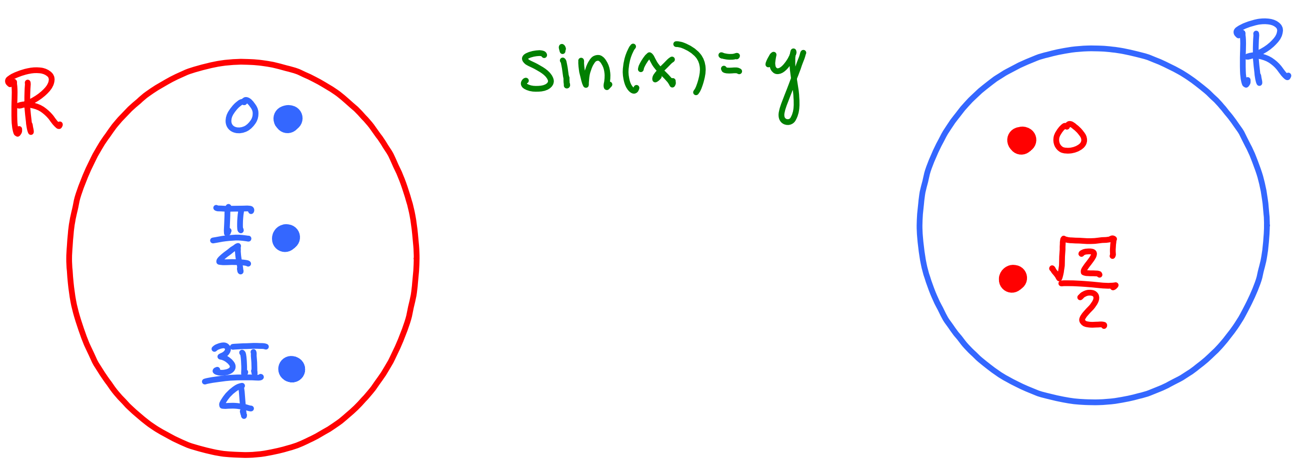
A FUNCTION, BY DEFINITION, CANNOT TAKE ONE x TO TWO DIFFERENT y'S.
IT'S GRAPH MUST PASS THE VERTICAL LINE TEST.



NOT A FUNCTION

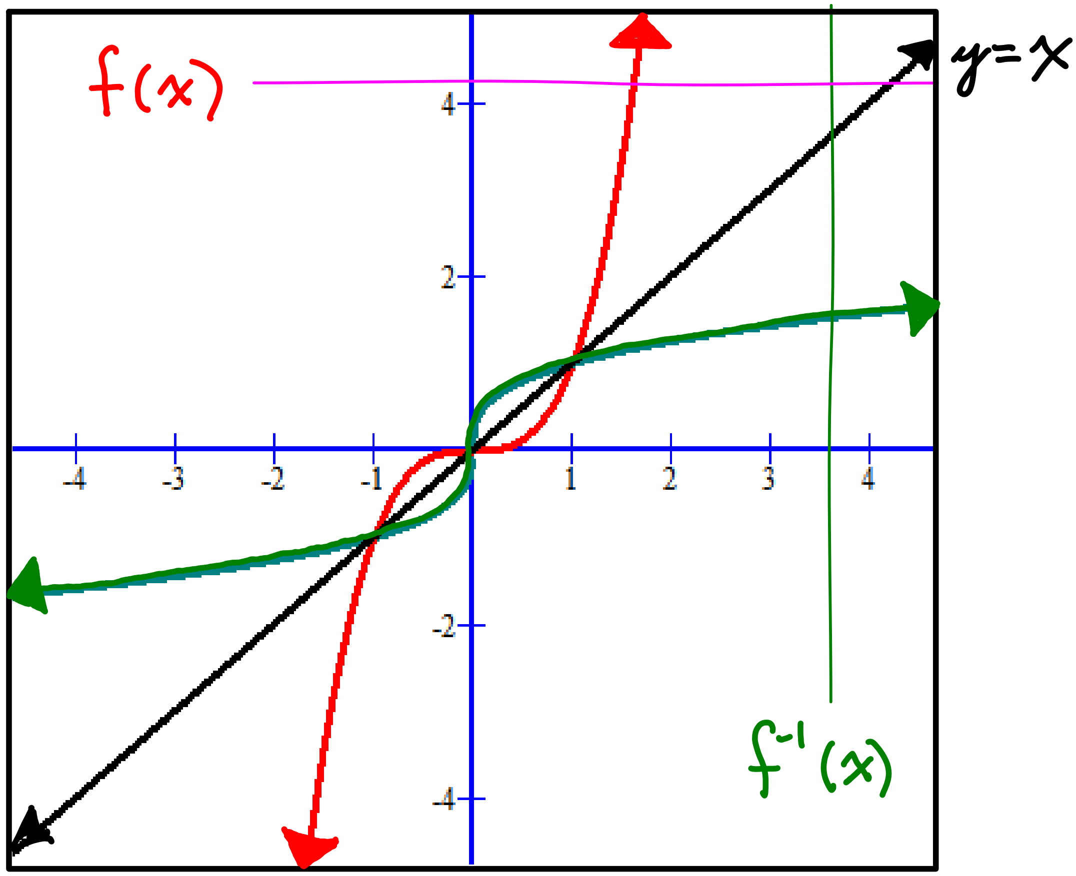
THIS IS A FUNCTION





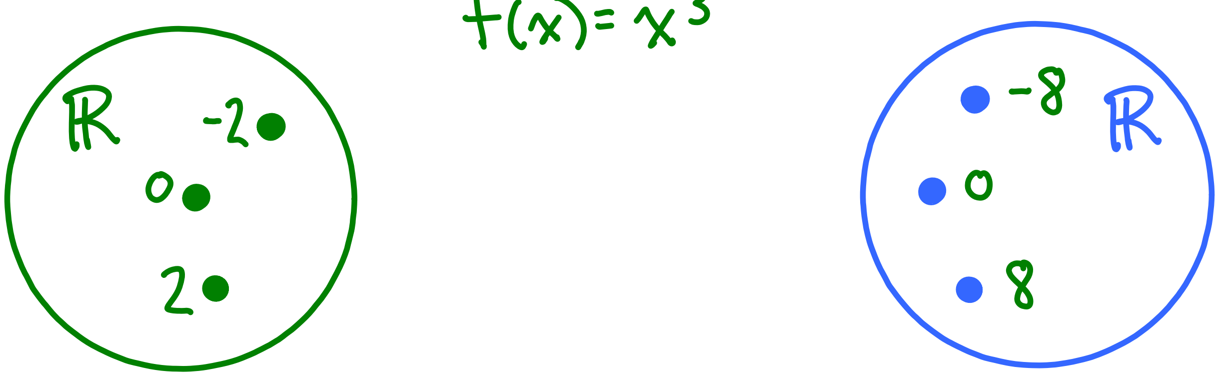
Some Functions are One-To-One

They pass the horizontal line test



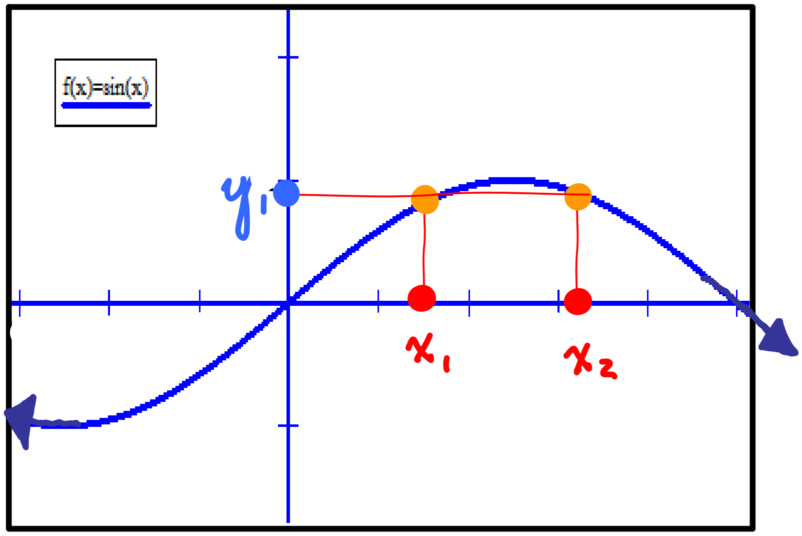
One-To-One Functions Have Inverse Functions

$f(x) = x^3$



We know exactly which x each y came from.

Trigonometric Functions Are not One-To-One



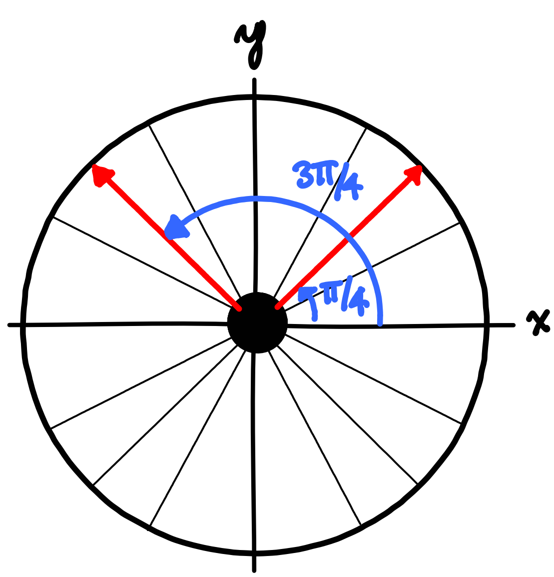
Did y_1 come from x_1 or x_2?

To solve for x in $\sin(x) = \frac{\sqrt{2}}{2}$
We must undo what sine did to x

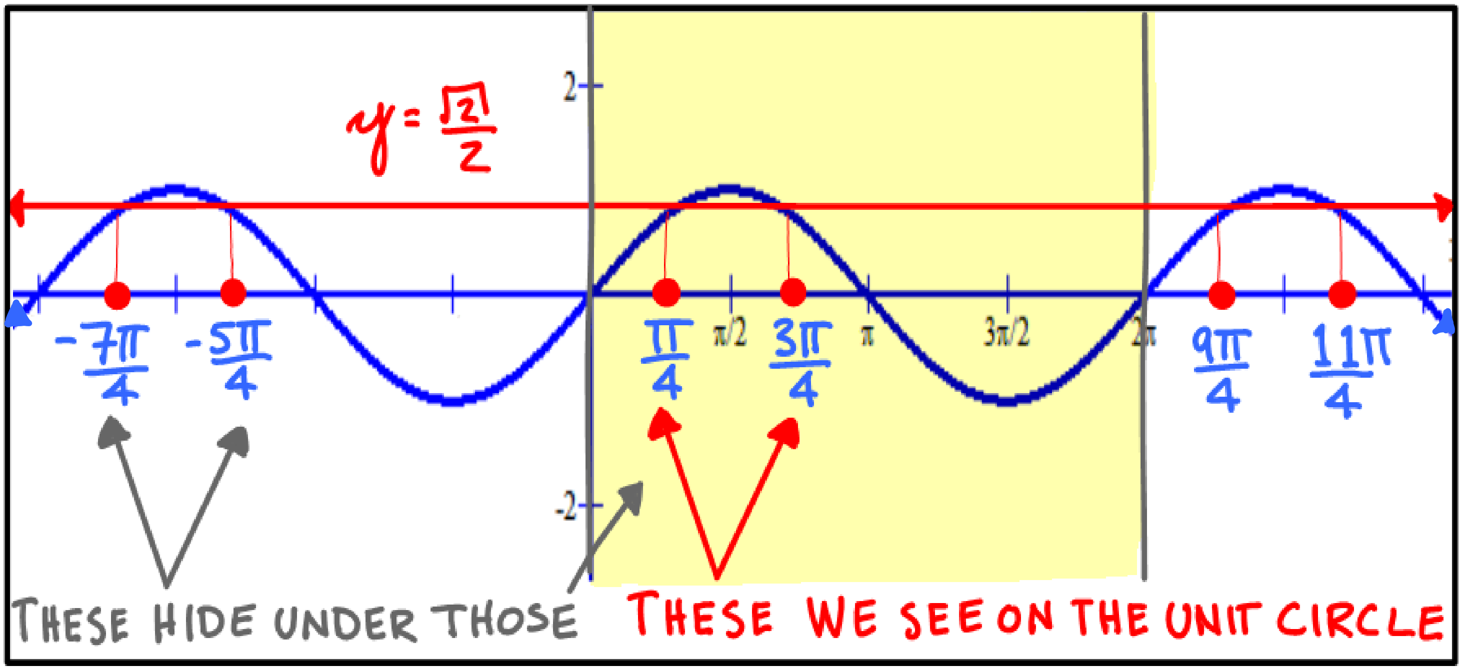
$$\sin(x) = \frac{\sqrt{2}}{2}$$

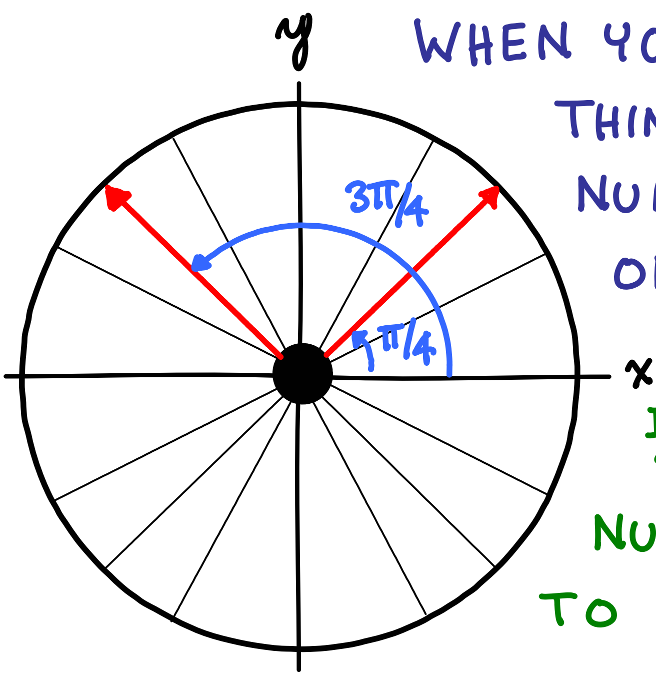
$$\Rightarrow x \in \left\{ \dots, -\frac{13\pi}{4}, -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \dots \right\}$$

$$= \{ \pi/4 + 2n\pi, 3\pi/4 + 2n\pi ; n \in \mathbb{Z} \}$$



$\sin(x) = \frac{\sqrt{2}}{2}$ HAS AN INFINITE NUMBER OF SOLUTIONS





WHEN YOU THINK OF THE UNIT CIRCLE THINK OF IT AS AN INFINITE NUMBER OF CIRCLES STACKED ON TOP OF EACH OTHER.

UNDERNEATH THE ANGLES $\frac{\pi}{4}$ AND $\frac{3\pi}{4}$ ARE AN INFINITE NUMBER OF ANGLES EQUIVALENT TO $\frac{\pi}{4}$ AND $\frac{3\pi}{4}$

WHAT IF WE WANT TO SOLVE $\sin(x) = 0.79346$?

LET'S DESIGN A FUNCTION THAT GIVES US ONE POSSIBLE x, THAT A CERTAIN y CAME FROM.

WE'LL CALL IT **ARCSINE** OR **SINE INVERSE**

$$\sin^{-1}(x) \equiv \arcsin(x)$$

WE WANT TO DO THIS:

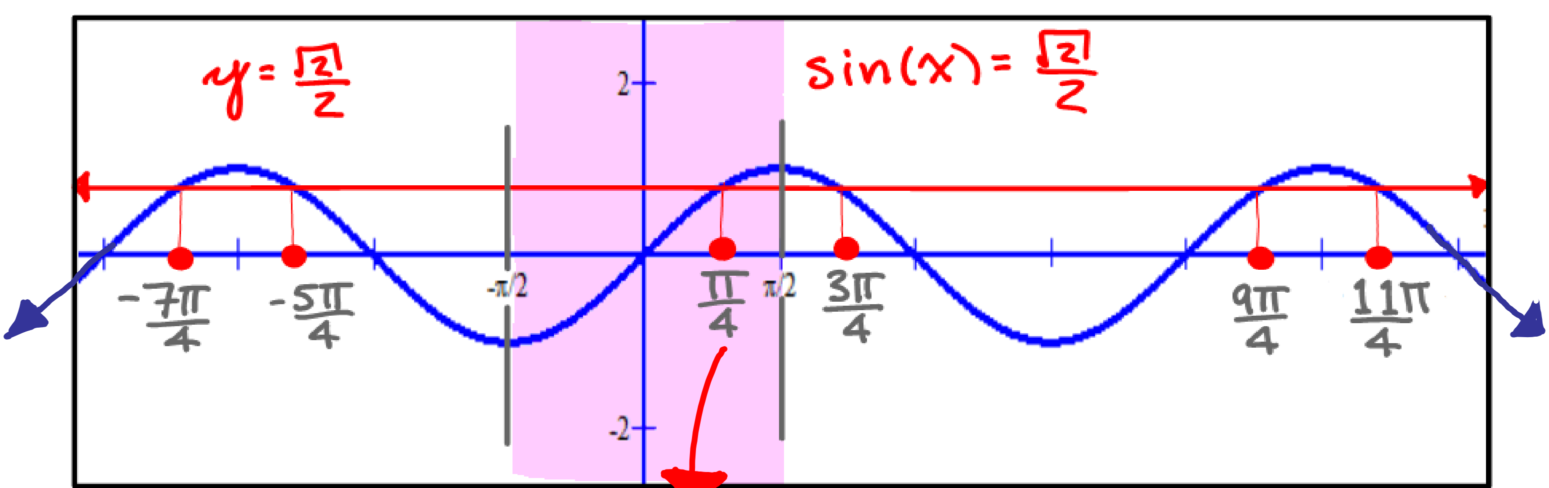
$$\sin(x) = 0.79346$$

$$\sin^{-1}(\sin(x)) = \sin^{-1}(0.79346)$$

$\Rightarrow \quad x = \sin^{-1}(0.79346) \leftarrow$ PLUG INTO CALCULATOR IT GIVES US A SOLUTION

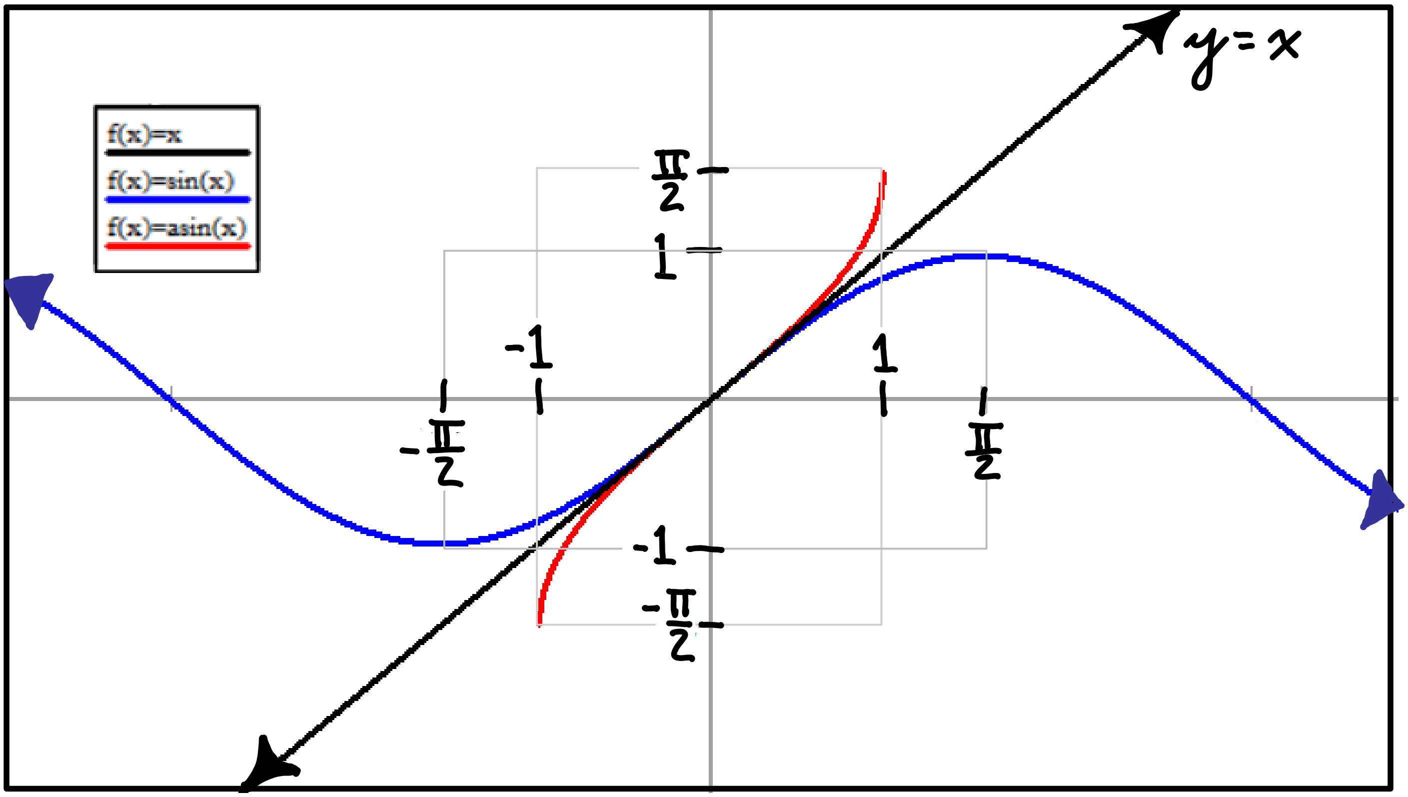
WE FIGURE OUT THE REST OURSELVES.

WE JUST NEED TO DECIDE WHICH x WE WANT THE CALCULATOR TO RETURN.



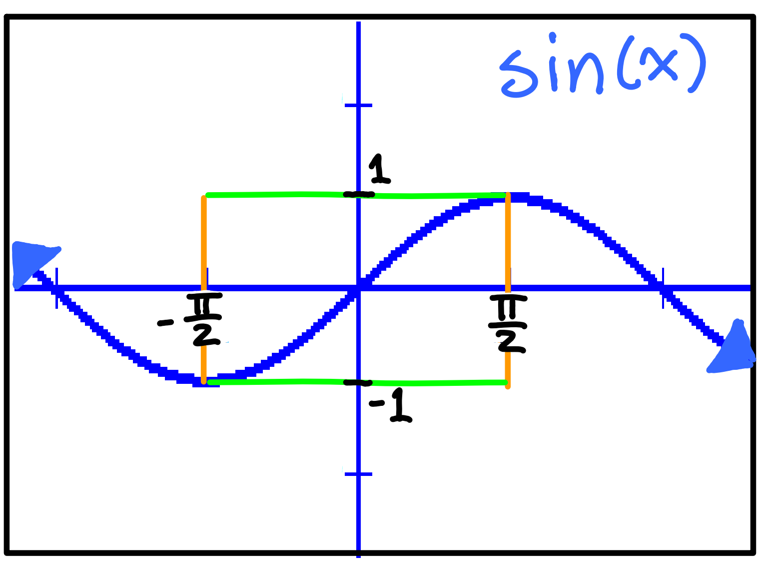
$$\sin\left(-\frac{7\pi}{4}\right) = \sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{11\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

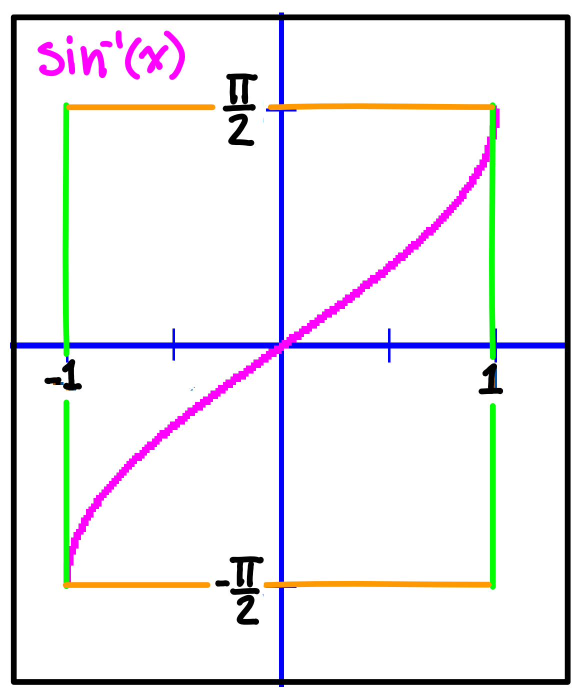
arcsin(x) ≡ asin(x) ≡ $\sin^{-1}(x)$

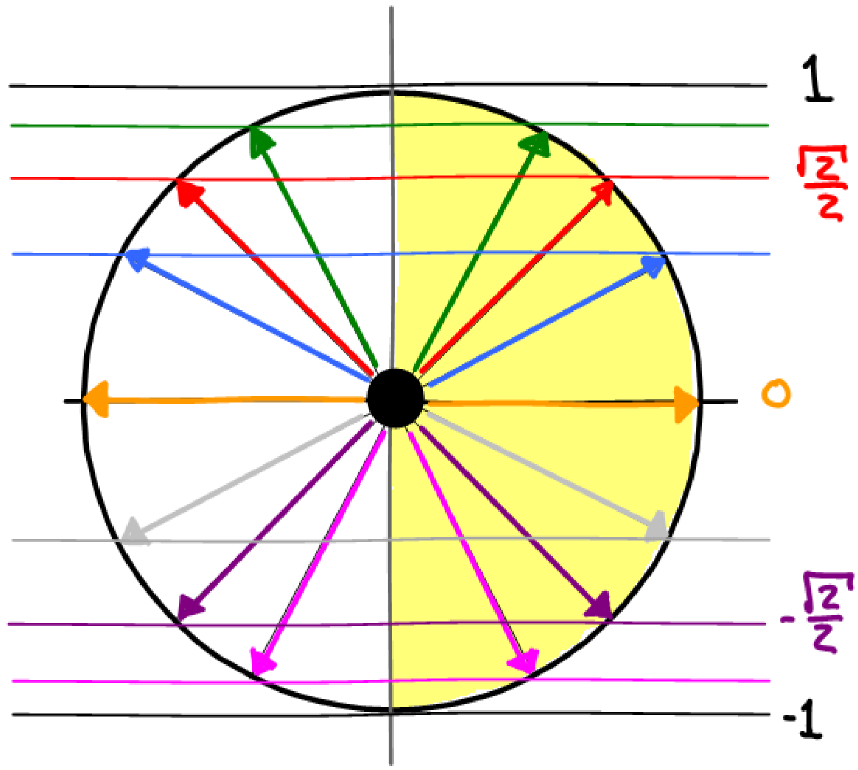


Domain and Range of Arcsine

sin(x) → [,] $\sin^{-1}(x)$ → [,] ⊂ ℝ







RANGE OF ARCSINE

THE LEFT HAND SIDE OF THE UNIT CIRCLE
IS A MIRROR IMAGE OF
THE RIGHT HAND SIDE OF THE UNIT CIRCLE
AS FAR AS SINE IS CONCERNED

ArcSine gives you an angle on
The Right Half of the Unit Circle.

FOR WHICH VALUES OF x DOES:

- $\sin^{-1}(\sin(x)) = x$?

$= x$ if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

otherwise you get a different # such that $\sin(\#) = \sin(x)$

- $\sin(\sin^{-1}(x)) = x$?

$= x$ if $x \in [-1, 1]$

Otherwise it is undefined.

WHAT ARE THE QUESTIONS ASKING?

WHEN DO YOU GET BACK THE SAME NUMBER THAT YOU PUT IN?

WHAT DO YOU KNOW THAT YOU CAN RELATE TO THE QUESTION?

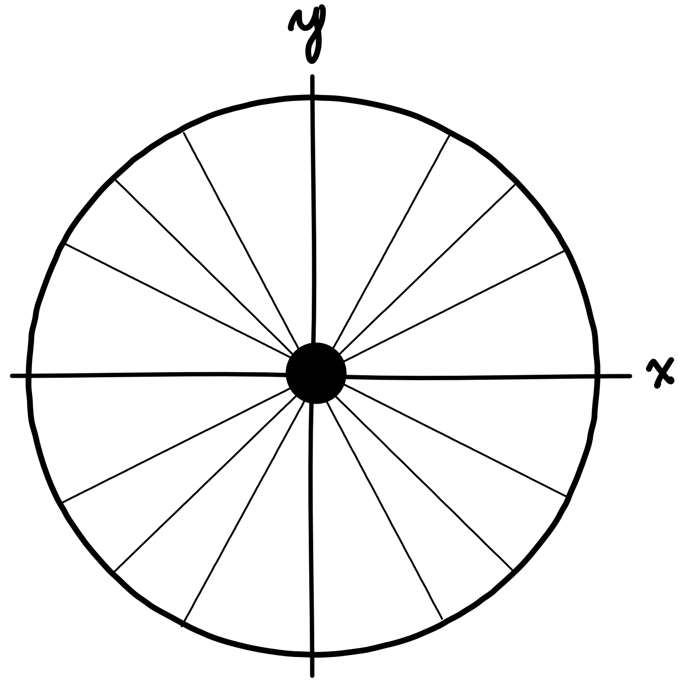
DOMAINS AND RANGES OF EACH FUNCTION.

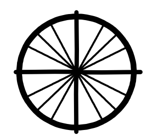
ANSWER

$\forall x \in \mathbb{R}$ $\sin(x) \in [-1, 1]$

If $y \in [-1, 1]$ then $\sin^{-1}(y) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

otherwise $\sin^{-1}(y)$ is undefined.





Examples:

$\sin^{-1}\left(\sin\left(-\frac{2\pi}{3}\right)\right) =$

$\sin^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) =$

$\sin^{-1}\left(\sin\left(-\frac{7\pi}{3}\right)\right) =$

$\sin^{-1}\left(\sin\left(-\frac{13\pi}{4}\right)\right) =$

$\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) =$

$\sin^{-1}\left(\sin\left(\pi\right)\right) =$

$\sin^{-1}\left(\sin\left(\frac{37\pi}{4}\right)\right) =$

$\sin^{-1}\left(\sin\left(\frac{16\pi}{3}\right)\right) =$

$\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) =$

$\sin^{-1}\left(\sin\left(\frac{23\pi}{3}\right)\right) =$

$\sin^{-1}\left(\sin\left(\frac{13\pi}{6}\right)\right) =$

$\sin^{-1}\left(\sin\left(-\frac{145\pi}{6}\right)\right) =$

MORE EXAMPLES: $\sin(\sin^{-1}(x))$

$\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) =$

$\sin\left(\sin^{-1}(2)\right) =$

$\sin\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) =$

$\sin\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right) =$

$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) =$

$\sin\left(\sin^{-1}(0)\right) =$

$\sin\left(\sin^{-1}(-2)\right) =$

$\sin\left(\sin^{-1}(-9)\right) =$

$\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) =$

$\sin\left(\sin^{-1}(1)\right) =$

FOR "UGLY" NUMBERS, USE A CALCULATOR

SET THE MODE TO RADIANS
WHEN USING YOUR CALCULATOR TO
FIND TRIG INVERSES

Losing and Adding Solutions

Example: Solve for x: $x^2 - 1 = 0$

$$x^2 - 1 = 0 \iff (x+1)(x-1) = 0$$

$$\iff \begin{cases} x+1=0 \\ x-1=0 \end{cases}$$

$$\iff x = \pm 1$$

Compare to this solution

$$x^2 - 1 = 0 \iff x^2 = 1$$

$$\iff \sqrt{x^2} = \sqrt{1}$$

$$\iff |x| = 1 \Rightarrow \begin{cases} x = 1 \\ x = -1 \end{cases}$$

$$\sqrt{x^2} \equiv |x|$$

$$(\sqrt{x})^2 \equiv \ ?$$

Consider restricting $x \geq 0$

EXAMPLE: SOLVE FOR x: $2\sin(x)=1$

$2\sin(x)=1 \Leftrightarrow \sin(x)=1/2$

$\Leftrightarrow$

COMPARE TO THIS SOLUTION

$2\sin(x)=1 \Leftrightarrow \sin(x)=1/2$

$\Leftrightarrow \sin^{-1}(\sin(x)) = \sin^{-1}(\frac{1}{2}) = \pi/6$

$\Rightarrow x = \pi/6$