Intro To Proving Identities

Reciprocal and Quotient Identities

$$\sec\theta \equiv \frac{1}{\cos\theta} \Leftrightarrow \cos\theta \equiv \frac{1}{\sec\theta}$$

$$\csc\theta \equiv \frac{1}{\sin\theta} \Leftrightarrow \sin\theta \equiv \frac{1}{\csc\theta}$$

$$\tan\theta \equiv \frac{1}{\cot\theta} \equiv \frac{\sin\theta}{\cos\theta} \equiv \frac{\sec\theta}{\csc\theta}$$

$$\cot\theta \equiv \frac{1}{\tan\theta} \equiv \frac{\cos\theta}{\sin\theta} \equiv \frac{\csc\theta}{\sec\theta}$$

Pythagorean Identities $x^2+y^2=r^2$

$$x^2+y^2=r^2 \Leftrightarrow \frac{x^2}{r^2}+\frac{y^2}{r^2}=\frac{r^2}{r^2} \Leftrightarrow \cos^2\theta+\sin^2\theta\equiv 1$$

$$x^2+y^2=r^2 \Rightarrow \frac{x^2}{x^2}+\frac{y^2}{x^2}=\frac{r^2}{x^2} \Leftrightarrow 1+\tan^2\theta\equiv\sec^2\theta$$

$$x^2+y^2=r^2 \Rightarrow \frac{x^2}{y^2}+\frac{y^2}{y^2}=\frac{r^2}{y^2} \Leftrightarrow \cot^2\theta+1=\csc^2\theta$$

Odd and Even Identities

$\sin(-\theta) \equiv -\sin\theta$

$\cos(-\theta) \equiv \cos\theta$

$\tan(-\theta) \equiv \dfrac{\sin(-\theta)}{\cos(-\theta)} \equiv \dfrac{-\sin\theta}{\cos\theta} \equiv -\tan\theta$

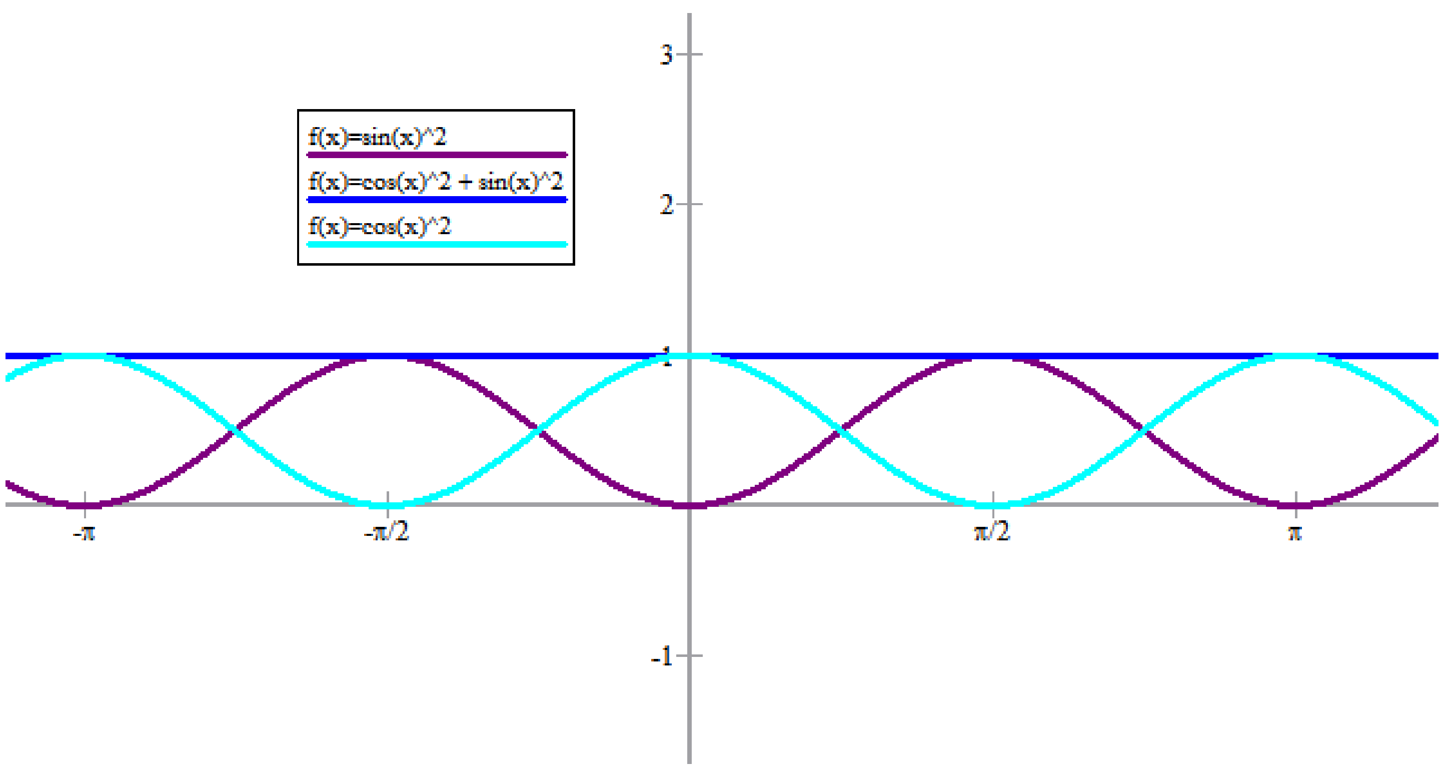
$\cot(-\theta) \equiv \dfrac{1}{\tan(-\theta)} \equiv \dfrac{1}{-\tan\theta} \equiv -\cot\theta$

$\sec(-\theta) \equiv \dfrac{1}{\cos(-\theta)} \equiv \dfrac{1}{\cos\theta} \equiv \sec\theta$

$\csc(-\theta) \equiv \dfrac{1}{\sin(-\theta)} \equiv \dfrac{1}{-\sin\theta} \equiv -\csc\theta$

Note: $1-\sin^2\theta \equiv 1^2-\sin^2\theta \equiv (1-\sin\theta)(1+\sin\theta)$

What Does "Identity" Mean?



Simplify $\frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta} \cdot \cos\theta = \sin\theta$

The Conjugate Trick

The conjugate of a+b is a-b and vice versa.

Prove the identity $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$

$$\frac{\cos\theta}{1+\sin\theta} = \left(\frac{\cos\theta}{1+\sin\theta}\right)\left(\frac{1-\sin\theta}{1-\sin\theta}\right) = \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} = \frac{1-\sin\theta}{\cos\theta}$$

Could't I just cross multiply? Sometimes.

Notation: $\Rightarrow$ $\Leftarrow$ and $\Leftrightarrow$

$A \Rightarrow B$ "A implies B" or "If A then B"

$A \Leftarrow B$ "B implies A" or "If B then A"

$A \Leftrightarrow B$ "A if and only if B"

If and only if $\rightarrow$ iff

ALTERNATE PROOF OF $\dfrac{\cos\theta}{1+\sin\theta} \equiv \dfrac{1-\sin\theta}{\cos\theta}$

$\sin^2\theta + \cos^2\theta \equiv 1 \qquad \theta \in \mathbb{R}$

$\Leftrightarrow \cos^2\theta = 1 - \sin^2\theta = (1-\sin\theta)(1+\sin\theta)$

$\Rightarrow \dfrac{\cos^2\theta}{1+\sin\theta} = 1 - \sin\theta \qquad \theta \in \mathbb{R} \setminus \{-\frac{\pi}{2} + 2n\pi ; n \in \mathbb{Z}\} \subset \mathbb{R}$

$\Rightarrow \dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta} \qquad \theta \in \mathbb{R} \setminus \{-\frac{\pi}{2} + n\pi ; n \in \mathbb{Z}\}$

$\subset \mathbb{R} \setminus \{-\frac{\pi}{2} + 2n\pi ; n \in \mathbb{Z}\}$

PROVE $$\frac{1+\sin\theta}{\sin\theta} + \frac{\cot\theta - \frac{1}{\sec\theta}}{\cos\theta} \equiv 2\csc\theta$$

$$\frac{1+\sin\theta}{\sin\theta} + \frac{\cot\theta - \frac{1}{\sec\theta}}{\cos\theta} \equiv \frac{1}{\sin\theta} + \frac{\sin\theta}{\sin\theta} + \frac{\frac{\cos\theta}{\sin\theta} - \cos\theta}{\cos\theta}$$

$$\equiv \csc\theta + \frac{\sin\theta}{\sin\theta} + \frac{\left(\frac{1}{\sin\theta} - 1\right)\cancel{\cos\theta}}{\cancel{\cos\theta}}$$

$$\equiv \csc\theta + 1 + \csc\theta - 1 \equiv 2\csc\theta$$

PROVE $\csc\theta\tan\theta \equiv \sec\theta$

$$\csc\theta\tan\theta \equiv \csc\theta\,\frac{\sin\theta}{\cos\theta} \equiv \csc\theta\sin\theta\left(\frac{1}{\cos\theta}\right)$$

$$\equiv \frac{\cancel{\csc\theta}}{\cancel{\csc\theta}}\sec\theta \equiv \sec\theta$$

Sometimes it's better to work with $\sec\theta$ and $\csc\theta$ rather than $\sin\theta$ and $\cos\theta$.

$$\csc\theta\tan\theta \equiv \csc\theta\,\frac{\sec\theta}{\csc\theta} \equiv \sec\theta$$

PROVE $-\sin^2(-\theta)-\cos^2(-\theta) \equiv -1$

$$-\sin^2(-\theta)-\cos^2(-\theta) \equiv -\left[\sin(-\theta)\sin(-\theta)\right]-\left[\cos(-\theta)\cos(-\theta)\right]$$

$$\equiv -(-\sin\theta)(-\sin\theta)-\cos\theta\cos\theta$$

$$\equiv -\sin^2\theta-\cos^2\theta \equiv -\left(\sin^2\theta+\cos^2\theta\right) \equiv -1$$

2nd solution:

$$\sin^2\theta+\cos^2\theta \equiv 1 \quad \theta\in\mathbb{R}$$

$$\Leftrightarrow \sin^2(-\theta)+\cos^2(-\theta) \equiv 1 \quad \text{since} \quad \theta\in\mathbb{R}\Leftrightarrow -\theta\in\mathbb{R}$$

$$\Leftrightarrow -\sin^2(-\theta)-\cos^2(-\theta) \equiv -1$$

PROVE $\quad \dfrac{\cos^2(-\theta)-\sin^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} \equiv \sin\theta-\cos\theta$

$$\frac{\cos^2(-\theta)-\sin^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} \equiv \frac{\cos^2\theta-\sin^2\theta}{-\sin\theta-\cos\theta}$$

$$\equiv \frac{(\cos\theta-\sin\theta)\cancel{(\cos\theta+\sin\theta)}}{-\cancel{(\sin\theta+\cos\theta)}}$$

$$\equiv \frac{\cos\theta-\sin\theta}{-1} \equiv \sin\theta-\cos\theta$$

PROVE $$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} \equiv 2\csc\theta$$

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \left(\frac{\sin\theta}{1+\cos\theta}\right)\frac{\sin\theta}{\sin\theta} + \left(\frac{1+\cos\theta}{\sin\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$= \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1 + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}}$$

$$= \frac{2}{\sin\theta} = 2\csc\theta$$

PROVE $$\frac{\tan\theta + \cot\theta}{\sec\theta \csc\theta} \equiv 1$$

$$\frac{\tan\theta + \cot\theta}{\sec\theta \csc\theta} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}}$$

$$= \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)\left(\frac{\cos\theta \sin\theta}{1}\right)$$

$$= \sin^2\theta + \cos^2\theta = 1$$

Prove $\sin(\tan^{-1}(t)) \equiv \dfrac{t}{\sqrt{1+t^2}}$

Preliminary Analysis:

Define $\theta := \tan^{-1}(t) \in (-\pi/2, \pi/2)$

Then $\tan\theta = t$ and $\sin(\tan^{-1}(t)) = \sin\theta$

Therefore,

if $t \geq 0$ then $t = \tan\theta \geq 0 \Rightarrow \theta \in [0, \pi/2) \Rightarrow \sin\theta \geq 0$ and

if $t < 0$ then $t = \tan\theta < 0 \Rightarrow \theta \in (-\pi/2, 0) \Rightarrow \sin\theta < 0$

thus, $t \geq 0 \Rightarrow \sin\theta \geq 0$ and $t < 0 \Rightarrow \sin\theta < 0$

So, for a geometric solution, let $\tan\theta = t = y/x$ for some y and some x.

If we choose $y := t$ and $x := 1$, then $y/x = t$ as desired

and $r^2 = x^2 + y^2 \Rightarrow r = \pm\sqrt{x^2+y^2} = \pm\sqrt{1^2+t^2} = \pm\sqrt{1+t^2}$.

Therefore, $\sin\theta = y/r = t/\pm\sqrt{1+t^2}$

To resolve the $\pm$, consider the preliminary analysis:

$t \geq 0 \Rightarrow \sin\theta \geq 0 \Leftrightarrow \sin\theta = \dfrac{t}{\sqrt{1+t^2}} \geq 0$ since $t \geq 0$

$t < 0 \Rightarrow \sin\theta < 0 \Leftrightarrow \sin\theta = \dfrac{t}{\sqrt{1+t^2}} < 0$ since $t < 0$

Thus, we do not want to include the minus sign. So,

$$\sin\theta = \boxed{\sin(\tan^{-1}(t)) \equiv \frac{t}{\sqrt{1+t^2}}}$$

Or, for a trigonometric solution, consider the pythagorean identity

$$\cot^2\theta + 1 \equiv \csc^2\theta \iff \frac{1}{\tan^2\theta} + 1 \equiv \frac{1}{\sin^2\theta}$$

$$\iff \sin^2\theta \equiv \frac{1}{\frac{1}{\tan^2\theta}+1} = \frac{1}{\frac{1}{t^2}+1} = \frac{1}{\frac{1}{t^2}+\frac{t^2}{t^2}}$$

$$= \frac{1}{\frac{1+t^2}{t^2}} = \frac{t^2}{1+t^2}$$

$$\iff \sin\theta = \pm\sqrt{\frac{t^2}{1+t^2}} = \frac{\pm|t|}{\sqrt{1+t^2}}$$

Note: $\sqrt{x^2} = |x|$

Again, to resolve the $\pm$ and the absolute value, consider the results from the preliminary analysis:

$t \geq 0 \Rightarrow \sin\theta \geq 0 \iff \sin\theta = \frac{t}{\sqrt{1+t^2}} \geq 0$ since $t \geq 0$

$t < 0 \Rightarrow \sin\theta < 0 \iff \sin\theta = \frac{t}{\sqrt{1+t^2}} < 0$ since $t < 0$

Thus, we do not want to include the minus sign. So,

$$\sin\theta = \boxed{\sin(\tan^{-1}(t)) \equiv \frac{t}{\sqrt{1+t^2}}}$$

STRATEGY

- KEEP IN MIND YOUR TARGET FORM
- DON'T SUBSTITUTE OR FACTOR "JUST BECAUSE YOU CAN."
- HOW MANY DIFFERENT KINDS OF TRIG FUNCTIONS ARE ON EACH SIDE OF THE IDENTITY?

  YOU ONLY NEED 2 TRIG FUNCTIONS.
- HOW MANY TERMS ARE ON EACH SIDE OF THE IDENTITY?

  FIND WAYS TO CONVERT BETWEEN SUMS AND PRODUCTS.
- IS THERE A FRACTION ON EACH SIDE OF THE IDENTITY?
- WHAT DOES THE LEFT HAND SIDE HAVE/NOT HAVE IN COMMON WITH THE RIGHT HAND SIDE?
- REDUCING THE BIG SIDE TO THE SMALLER SIDE IS OFTEN EASIER.
- WILL CONVERTING TO SINES AND COSINES HELP?
- USE "$\Rightarrow$" and "$\Leftrightarrow$" for a less restricted approach to proving identities.

PROVE $\dfrac{1+\tan\theta}{1-\tan\theta} \equiv \dfrac{\cot\theta+1}{\cot\theta-1}$

$$\frac{1+\tan\theta}{1-\tan\theta} = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{\cot\theta}{\cot\theta}\right)$$

$$= \frac{\cot\theta+\tan\theta\cot\theta}{\cot\theta-\tan\theta\cot\theta}$$

$$= \frac{\cot\theta+\frac{\tan\theta}{\tan\theta}}{\cot\theta-\frac{\tan\theta}{\tan\theta}} = \frac{\cot\theta+1}{\cot\theta-1}$$

Or convert to $\sin\theta$ and $\cos\theta$ and have a longer solution.

PROVE $\dfrac{\sin^3\theta + \cos^3\theta}{1 - 2\cos^2\theta} = \dfrac{\sec\theta - \sin\theta}{\tan\theta - 1}$

NOTE: $1 - 2\cos^2\theta = 1 - \cos^2\theta - \cos^2\theta$

Try working with each side separately

Left side:

$$\frac{\sin^3\theta + \cos^3\theta}{1 - 2\cos^2\theta}$$

$$= \frac{\sin^3\theta + \cos^3\theta}{\underbrace{1 - \cos^2\theta}_{} - \cos^2\theta}$$

$$= \frac{\sin^3\theta + \cos^3\theta}{\sin^2\theta - \cos^2\theta}$$

$$= \frac{\sin^3\theta + \cos^3\theta}{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}$$

Right side:

$$\frac{\sec\theta - \sin\theta}{\tan\theta - 1}$$

$$= \frac{\dfrac{1}{\cos\theta} - \sin\theta\left(\dfrac{\cos\theta}{\cos\theta}\right)}{\dfrac{\sin\theta}{\cos\theta} - \left(\dfrac{\cos\theta}{\cos\theta}\right)}$$

$$= \frac{\dfrac{1 - \sin\theta\cos\theta}{\cos\theta}}{\dfrac{\sin\theta - \cos\theta}{\cos\theta}}$$

compare forms to help you factor

$$= \frac{1 - \sin\theta\cos\theta}{\sin\theta - \cos\theta} =$$

$$= \frac{\cancel{(\sin\theta + \cos\theta)}(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)}{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}} = \frac{\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta}{\sin\theta - \cos\theta}$$

PROVE $\dfrac{\cot\theta}{1-\tan\theta} + \dfrac{\tan\theta}{1-\cot\theta} \equiv 1+\tan\theta+\cot\theta$

$$\frac{\cot\theta}{1-\tan\theta} + \frac{\tan\theta}{1-\cot\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}} + \frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}}$$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}} + \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\sin\theta}-\frac{\cos\theta}{\sin\theta}} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}} + \frac{\frac{\sin\theta}{\cos\theta}}{\frac{-(\cos\theta-\sin\theta)}{\sin\theta}}$$

$$= \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{(\cos\theta-\sin\theta)} - \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{(\cos\theta-\sin\theta)}$$

$$= \frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}\left(\frac{\cos\theta}{\cos\theta}\right) - \frac{\sin^2\theta}{\cos\theta(\cos\theta-\sin\theta)}\left(\frac{\sin\theta}{\sin\theta}\right)$$

$$= \frac{\cos^3\theta-\sin^3\theta}{\cos\theta\sin\theta(\cos\theta-\sin\theta)}$$

Remember your target $1+\tan\theta+\cot\theta$

$$= \frac{\cancel{(\cos\theta-\sin\theta)}(\cos^2\theta+\cos\theta\sin\theta+\sin^2\theta)}{\cos\theta\sin\theta\cancel{(\cos\theta-\sin\theta)}}$$

$$= \frac{\cos^2\theta}{\cos\theta\sin\theta} + \frac{\cos\theta\sin\theta}{\cos\theta\sin\theta} + \frac{\sin^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta} + 1 + \frac{\sin\theta}{\cos\theta}$$

$$= 1+\tan\theta+\cot\theta$$

There must be a shorter way to prove this!

Shorter proofs are worth extra credit!

NOTE ON NOTATION

LEARN TO DEVELOPE YOUR OWN NOTATION WHEN IT IS USEFUL.

SOMETIMES I WRITE $\$(x)$ FOR $\sin(x)$ AND
$¢(x)$ FOR $\cos(x)$
JUST TO SAVE TIME WRITING.

OF COURSE I HAVE TO STATE THIS AT THE BEGINNING OF MY WORK.

DEFINE $\$(x) := \sin(x)$ AND $¢(x) := \cos(x)$

THE ":=" SYMBOL IS OFTEN USED TO MEAN "IS DEFINED TO BE."

DON'T FORGET THAT YOU CAN USE THE DEFINITIONS BASED ON A CIRCLE,

$\tan\theta = y/x \qquad \cos\theta = x/r$ etc.

THIS MAY REDUCE WRITING AND PROVIDE EXTRA PERSPECTIVE.
