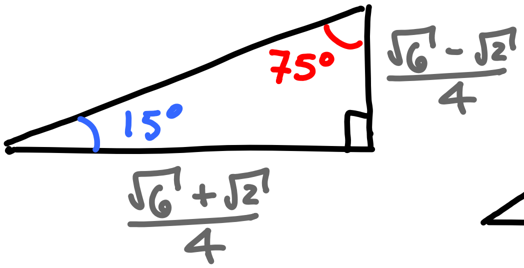
SUM AND DIFFERENCE

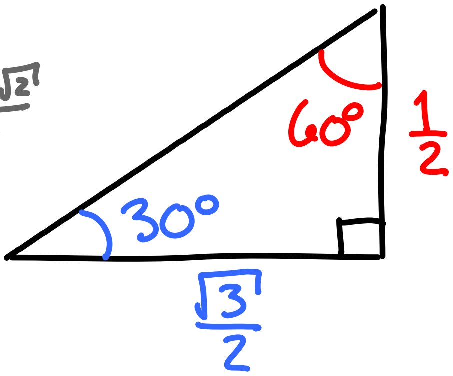
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

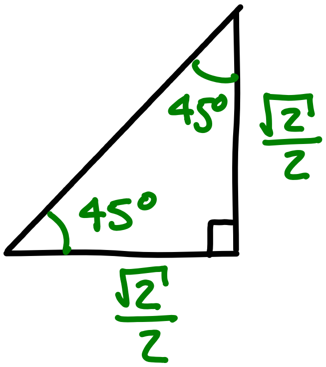
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos(\alpha+(-\beta))$$
$$= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(15^\circ) = \cos(45^\circ-30^\circ)$$
$$= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ)$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$





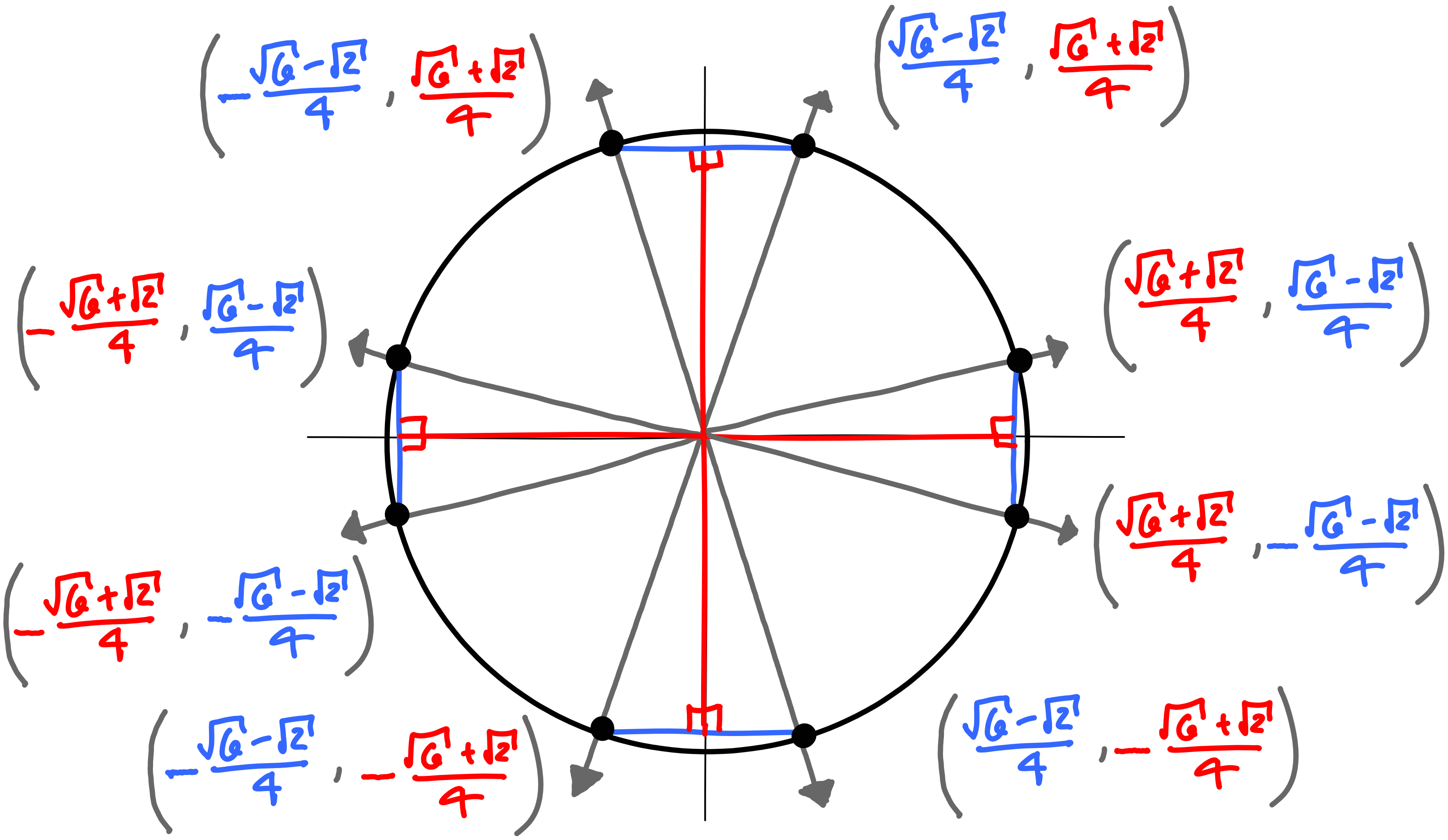


$$\sin(15^\circ) = \sin(45^\circ-30^\circ)$$
$$= \sin(45^\circ)\cos(30^\circ) - \cos(45^\circ)\sin(30^\circ)$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos(75^\circ) = \cos(45^\circ + 30^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4} = \sin(15^\circ)$$

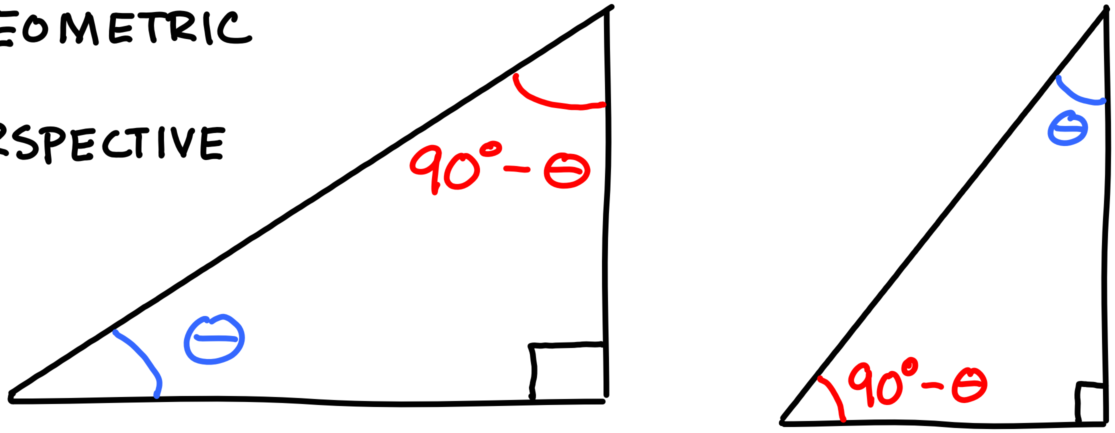
$$\sin(75^\circ) = \sin(45^\circ + 30^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4} = \cos(15^\circ)$$

EXTENDED UNIT CIRCLE



COFUNCTION IDENTITIES

GEOMETRIC PERSPECTIVE



COFUNCTION ALGEBRA

$\sin(\theta) = \cos(90° - \theta) = \cos(\pi/2 - \theta)$

$\cos(\theta) = \sin(90° - \theta) = \sin(\pi/2 - \theta)$

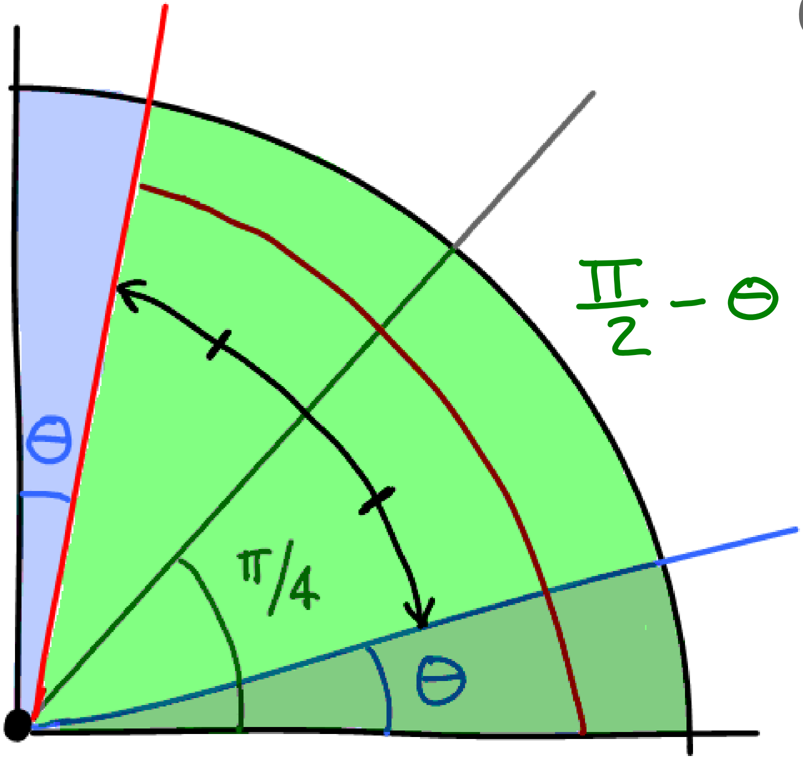
$\tan(\theta) = \cot(90° - \theta) = \cot(\pi/2 - \theta)$

$\cot(\theta) = \cot(90° - \theta) = \cot(\pi/2 - \theta)$

$\sec(\theta) = \csc(90° - \theta) = \csc(\pi/2 - \theta)$

$\csc(\theta) = \csc(90° - \theta) = \csc(\pi/2 - \theta)$

MORE COFUNCTION GEOMETRY

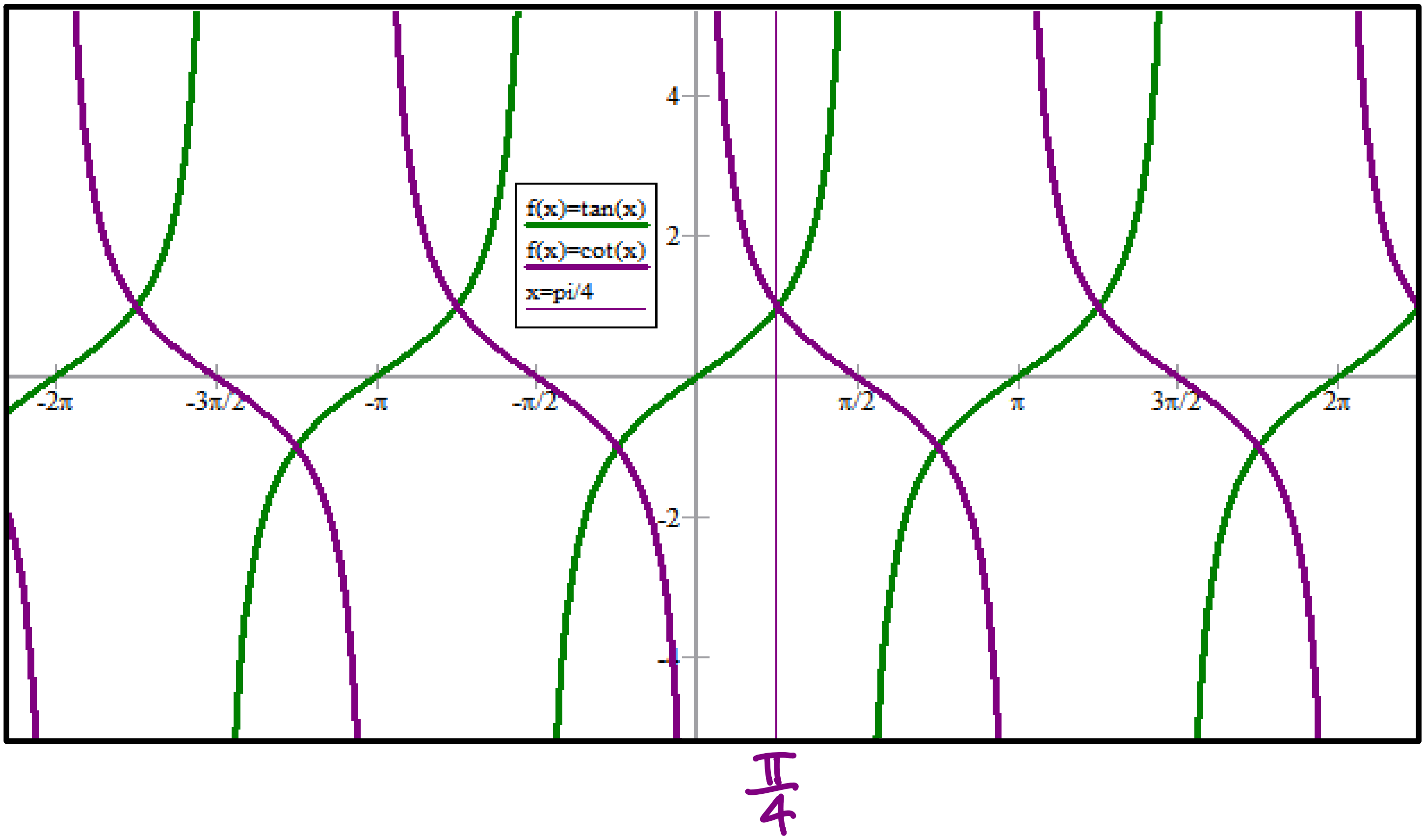


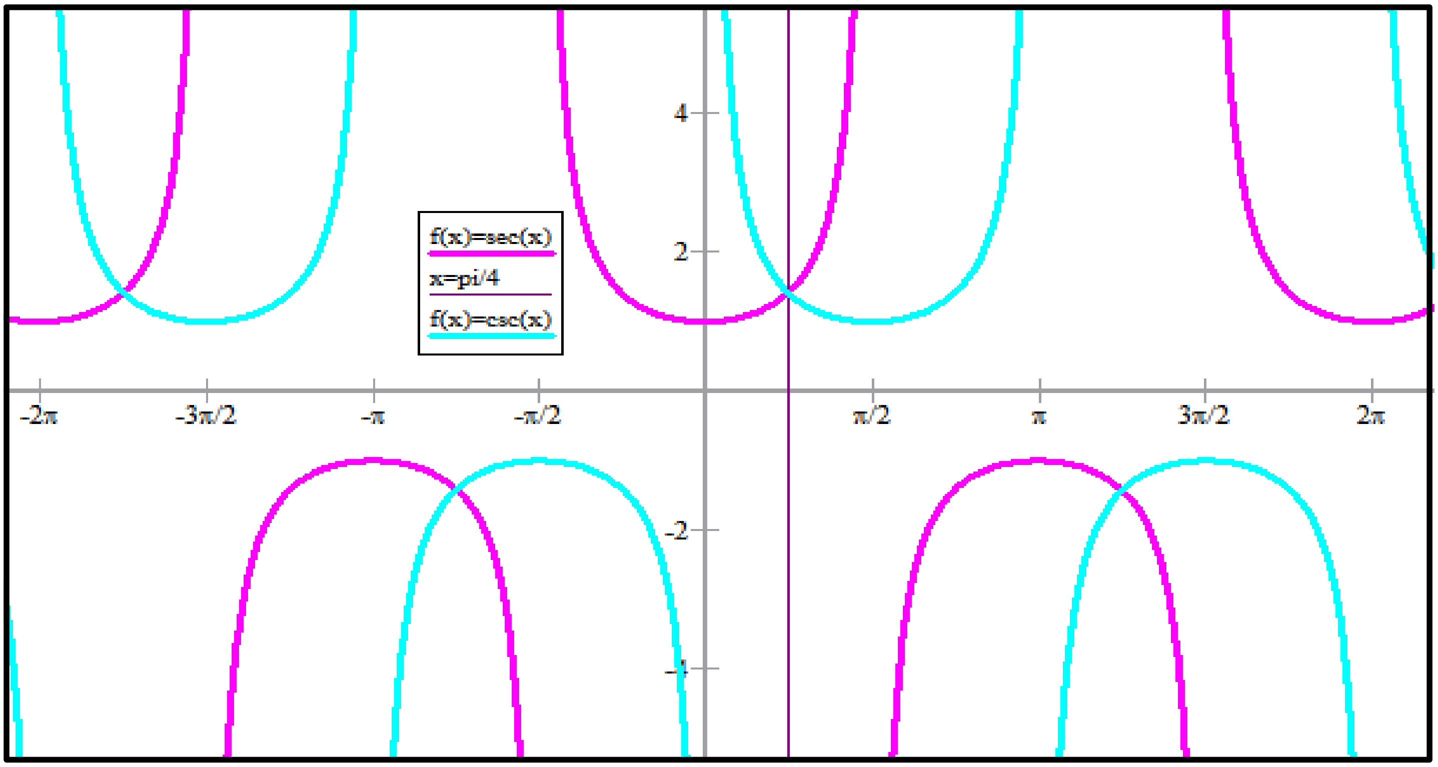
$\theta \longleftrightarrow \frac{\pi}{2} - \theta$

FLIP OVER π/4

GRAPHICAL REPRESENTATION OF COFUNCTIONS

SYMMETRY ABOUT THE VERTICAL LINE $x = \frac{\pi}{4}$

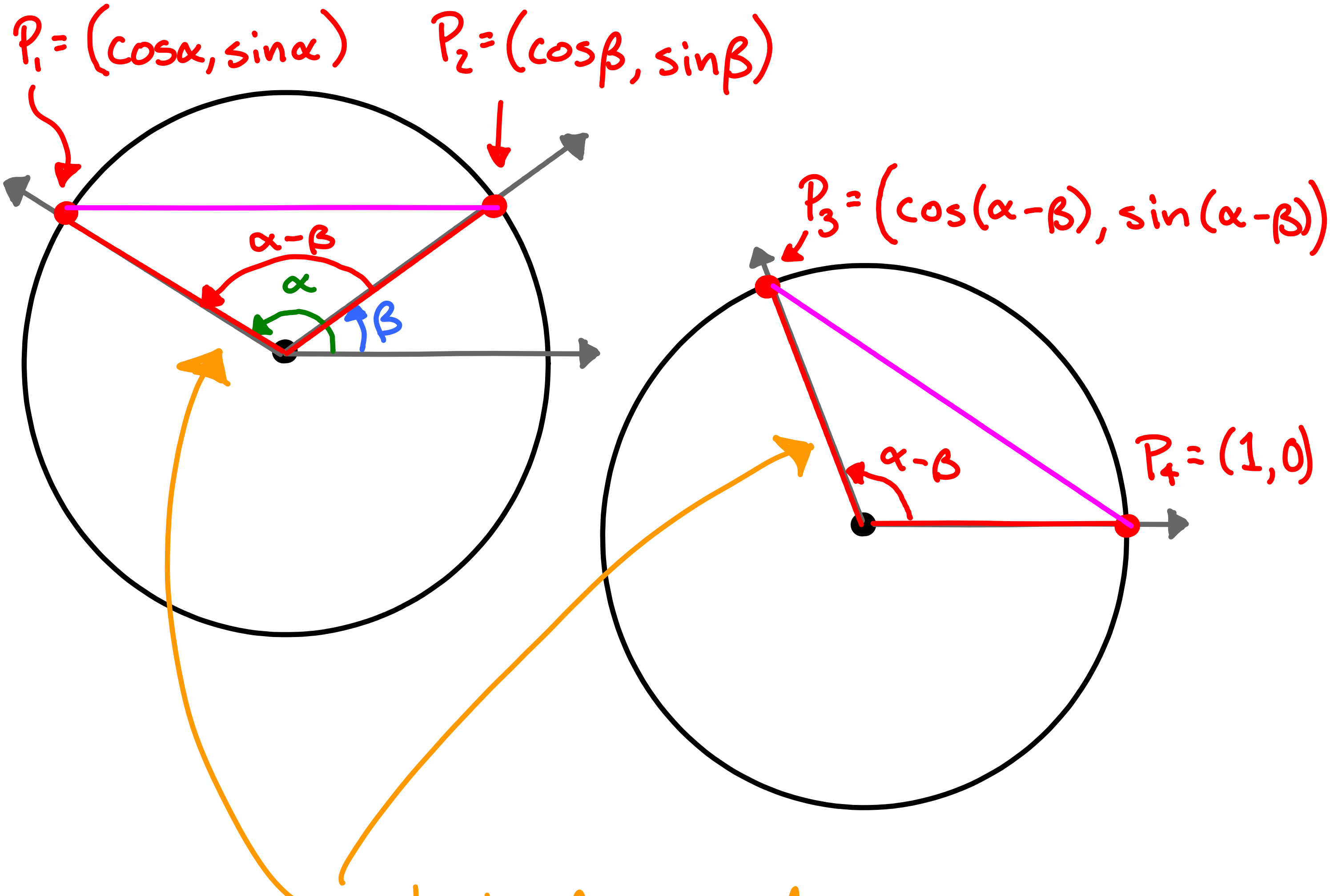




$\frac{\pi}{4}$

PROVE $\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$

START WITH THE UNIT CIRCLE.



Identical Triangles.

Pink sides are the same length.

These lengths relate

$\cos\alpha$, $\sin\alpha$, $\cos\beta$, and $\sin\beta$

to $\cos(\alpha-\beta)$ and $\sin(\alpha-\beta)$.

DISTANCE FORMULA

IF $P_1=(x_1,y_1)$ AND $P_2=(x_2,y_2)$ THEN

$$D(P_1,P_2)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$P_1=(\cos\alpha, \cos\beta)$ $P_2=(\cos\beta, \sin\beta)$

$P_3=(\cos(\alpha-\beta), \sin(\alpha-\beta))$ $P_4=(1,0)$

$$\Rightarrow \begin{cases} D(P_1,P_2)=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} \\ D(P_3,P_4)=\sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \end{cases}$$

Now, we now that

$$D(P_1,P_2)=D(P_3,P_4)$$

$$\Leftrightarrow D^2(P_1,P_2)={}^2D(P_3,P_4)$$

$$\Leftrightarrow (\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2$$

$$\Leftrightarrow \cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$$

$$=\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)$$

1 1 1

$$\Leftrightarrow \cancel{2}-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta=\cancel{2}-2\cos(\alpha-\beta)$$

$$\Leftrightarrow \boxed{\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta)}$$

FIND $\cos(\alpha+\beta)$ USING THE $\cos(\alpha-\beta)$ IDENTITY

$$\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\begin{aligned}\Rightarrow \cos(\alpha+\beta) &= \cos(\alpha-(-\beta)) \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

DERIVE $\sin(\alpha\pm\beta)$ USING THE $\cos(\alpha\pm\beta)$ IDENTITY

$$\begin{aligned}\sin(\alpha\pm\beta) &= \cos\left(\frac{\pi}{2}-(\alpha\pm\beta)\right) = \cos\left(\left(\frac{\pi}{2}-\alpha\right)\mp\beta\right) \\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta \pm \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta \\ &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\end{aligned}$$

SIMPLIFY $\sin(2\theta)\sin(3\theta) - \cos(3\theta)\cos(2\theta)$

$$\sin(2\theta)\sin(3\theta) - \cos(3\theta)\cos(2\theta)$$

$$= -\left[\cos(3\theta)\cos(2\theta) - \sin(2\theta)\sin(3\theta)\right]$$

$$= -\left[\cos(3\theta)\cos(2\theta) - \sin(3\theta)\sin(2\theta)\right]$$

$$= -\cos(3\theta + 2\theta) = -\cos(5\theta)$$

EVALUATE $\cos\left[\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{3}{4}\right)\right]$

$$\cos\left[\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{3}{4}\right)\right] = \cos(\alpha+\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

where $\alpha := \sin^{-1}\left(\frac{1}{3}\right) \in (0, \pi/2) \Rightarrow \sin\alpha = 1/3$

$$\Rightarrow \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-1/9} = \sqrt{8/9} = \frac{2\sqrt{2}}{3}$$

and $\beta := \cos^{-1}\left(\frac{3}{4}\right) \in (0, \pi/2) \Rightarrow \cos\beta = 3/4$

$$\Rightarrow \sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-9/16} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

Therefore,

$$\cos\left[\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{3}{4}\right)\right] = \cos(\alpha+\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{2\sqrt{2}}{3}\cdot\frac{3}{4} - \frac{1}{3}\frac{\sqrt{7}}{4} = \boxed{\frac{6\sqrt{2}-\sqrt{7}}{12}}$$

WRITE TAN$(\alpha \pm \beta)$ IN TERMS OF TANα AND TANβ

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$$

$$= \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

$$= \left(\frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}\right)\left(\frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}\right)$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

TANGENT SUM AND DIFFERENCE IDENTITIES

$$\boxed{\text{TAN}(\alpha \pm \beta) = \frac{\text{TAN}\alpha \pm \text{TAN}\beta}{1 \mp \text{TAN}\alpha\,\text{TAN}\beta}}$$

PROVE $\quad \mathrm{TAN}(\theta + \pi/2) \equiv -\mathrm{COT}\,\theta$

A) USING SUM IDENTITIES

B) USING COFUNCTION IDENTITIES

A) $$\mathrm{TAN}(\theta + \pi/2) = \frac{\tan\theta + \tan(\pi/2)}{1 - \tan\theta\tan(\pi/2)}$$

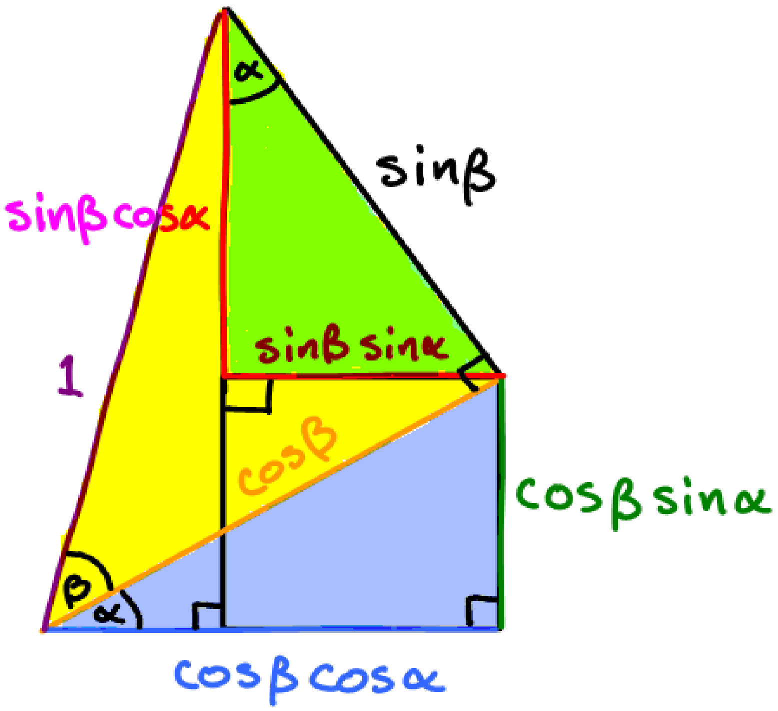
undefined ☹

$$\mathrm{TAN}(\theta + \pi/2) = \frac{\sin\theta\,\cancelto{0}{\cos\left(\frac{\pi}{2}\right)} + \cos\theta\,\cancelto{1}{\sin\left(\frac{\pi}{2}\right)}}{\cos\theta\,\cancelto{0}{\cos\left(\frac{\pi}{2}\right)} - \sin\theta\,\cancelto{1}{\sin\left(\frac{\pi}{2}\right)}}$$

$$= \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

B) $$\mathrm{TAN}(\theta + \pi/2) = \tan(\theta + \pi/2 - \pi)$$
$$= \tan(\theta - \pi/2)$$
$$= \tan[-(\pi/2 - \theta)]$$
$$= -\tan(\pi/2 - \theta)$$
$$= -\cot\theta$$

VERIFY THE PYTHAGOREAN THEOREM FOR THE 3 RIGHT TRIANGLES.



▲ $\sin^2\beta\sin^2\alpha + \sin^2\beta\cos^2\alpha = \sin^2\beta$

▲ $\cos^2\beta\cos^2\alpha + \cos^2\beta\sin^2\alpha = \cos^2\beta$

▲ $\cos^2\beta + \sin^2\beta = 1$

▲ + ▲ ⟺ $\sin^2\beta\sin^2\alpha + \sin^2\beta\cos^2\alpha$

$+ \cos^2\beta\cos^2\alpha + \cos^2\beta\sin^2\alpha$

$= \sin^2\beta + \cos^2\beta = 1$

⟺ $\sin^2\beta(\sin^2\alpha + \cos^2\alpha) + \cos^2\beta(\cos^2\alpha + \sin^2\alpha) = 1$

⟺ $\sin^2\beta + \cos^2\beta = 1$ ✓

WRITE $\cos(2\theta)$ IN TERMS OF $\cos\theta$ AND $\sin\theta$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\left.\begin{aligned} &= \cos^2\theta - \sin^2\theta \\ &= 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1 \end{aligned}\right\} = \cos(2\theta)$$