Product-to-sum Identities

These will be very useful in calculus

Write $\sin\alpha \sin\beta$ as a sum without products

$$\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

and

$$\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\Leftrightarrow \cos(\alpha-\beta) - \cos(\alpha+\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

$$= 2\sin\alpha\sin\beta$$

$$\Leftrightarrow \boxed{\sin\alpha\sin\beta \equiv \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]}$$

Write $\cos\alpha\cos\beta$ as a sum without products

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

and $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\Leftrightarrow \cos(\alpha-\beta) + \cos(\alpha+\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

$$= 2\cos\alpha\cos\beta$$

$$\Leftrightarrow \boxed{\cos\alpha\cos\beta \equiv \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]}$$

WRITE $\sin\alpha\cos\beta$ AS A SUM WITHOUT PRODUCTS

$$\sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

and $$\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\Leftrightarrow \sin(\alpha-\beta)+\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta + (\sin\alpha\cos\beta + \cos\alpha\sin\beta) = 2\sin\alpha\cos\beta$$

$$\Leftrightarrow \boxed{\sin\alpha\cos\beta \equiv \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]}$$

PRODUCT-TO-SUM IDENTITIES

$$\boxed{\begin{aligned} \sin\alpha\sin\beta &\equiv \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\ \cos\alpha\cos\beta &\equiv \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &\equiv \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \end{aligned}}$$

SUM-TO-PRODUCT IDENTITIES

THESE ARE USEFUL WHEN SOLVING EQUATIONS. TURN A SUM INTO A PRODUCT, THEN FACTOR LIKE YOU DID IN BASIC ALGEBRA.

WRITE $\sin\alpha \pm \sin\beta$ AS A PRODUCT

$$\sin\alpha\cos\beta \equiv \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$

$$\Leftrightarrow \quad \sin(\alpha-\beta)+\sin(\alpha+\beta) \equiv 2\sin\alpha\cos\beta$$

So, if $x=\alpha-\beta$ and $y=\alpha+\beta$, then

$$x+y=\alpha-\beta+\alpha+\beta=2\alpha \Rightarrow \alpha=\frac{x+y}{2} \text{ and}$$

$$x-y=\alpha-\beta-(\alpha+\beta)=-2\beta \Rightarrow \beta=-\frac{(x-y)}{2}$$

Thus, substituting x and y,

$$\sin(\alpha-\beta)+\sin(\alpha+\beta) \equiv 2\sin\alpha\cos\beta$$

$$\Leftrightarrow \quad \sin x+\sin y \equiv 2\sin\left(\frac{x+y}{2}\right)\cos\left(-\frac{(x-y)}{2}\right)$$

$$\Leftrightarrow \quad \sin x+\sin y \equiv 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\Leftrightarrow \quad \boxed{\sin\alpha+\sin\beta \equiv 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

Now, for $\sin\alpha - \sin\beta$, consider $\sin(-\beta) \equiv -\sin\beta$

So,

$$\sin\alpha + \sin\beta \equiv 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Leftrightarrow \quad \sin\alpha + \sin(-\beta) \equiv 2\sin\left(\frac{\alpha+(-\beta)}{2}\right)\cos\left(\frac{\alpha-(-\beta)}{2}\right)$$

$$\Leftrightarrow \quad \boxed{\sin\alpha - \sin\beta \equiv 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$$

WRITE $\cos\alpha + \cos\beta$ AS A PRODUCT

$$\cos\alpha\cos\beta \equiv \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\Leftrightarrow \quad \cos(\alpha-\beta)+\cos(\alpha+\beta) \equiv 2\cos\alpha\cos\beta$$

Again, substitute:

$x = \alpha-\beta$ and $y = \alpha+\beta$

$\Leftrightarrow \alpha = \frac{x+y}{2}$ and $\beta = -\frac{(x-y)}{2}$

$$\Leftrightarrow \quad \cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(-\frac{(x-y)}{2}\right)$$

$$\Leftrightarrow \quad \cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\Leftrightarrow \quad \boxed{\cos\alpha + \cos\beta \equiv 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

WRITE $\cos\alpha - \cos\beta$ AS A PRODUCT

$$\sin\alpha\sin\beta \equiv \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\Leftrightarrow \quad \cos(\alpha-\beta) - \cos(\alpha+\beta) \equiv 2\sin\alpha\sin\beta$$

Again, substitute:

$x = \alpha - \beta$ and $y = \alpha + \beta$

$\Leftrightarrow \alpha = \frac{x+y}{2}$ and $\beta = -\frac{(x-y)}{2}$

$$\Leftrightarrow \quad \cos x - \cos y \equiv 2\sin\left(\frac{x+y}{2}\right)\sin\left(-\frac{(x-y)}{2}\right)$$

$$\Leftrightarrow \quad \cos x - \cos y \equiv -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\Leftrightarrow \quad \boxed{\cos\alpha - \cos\beta \equiv -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)}$$

SUM-TO-PRODUCT IDENTITIES

$$\sin\alpha \pm \sin\beta \equiv 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\cos\alpha + \cos\beta \equiv 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha - \cos\beta \equiv -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

WRITE $\sin(5\theta)\cos(3\theta)$ AS A SUM

$$\sin\alpha\cos\beta \equiv \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$

$$\Leftrightarrow \quad \sin(5\theta)\cos(3\theta) = \frac{1}{2}\left[\sin(5\theta-3\theta)+\sin(5\theta+3\theta)\right]$$

$$= \boxed{\frac{1}{2}\left[\sin(2\theta)+\sin(8\theta)\right]}$$

WRITE $\sin(5\theta)+\sin(3\theta)$ AS A PRODUCT

$$\sin\alpha \pm \sin\beta \equiv 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\Leftrightarrow \quad \sin(5\theta)+\sin(3\theta) = 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$

$$= \boxed{2\sin(4\theta)\cos\theta}$$

PROVE $\dfrac{\sin(4\theta)+\sin(2\theta)}{\cos(4\theta)+\cos(2\theta)} \equiv \tan(3\theta)$

$$\sin\alpha \pm \sin\beta \equiv 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\cos\alpha + \cos\beta \equiv 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Leftrightarrow \frac{\sin(4\theta)+\sin(2\theta)}{\cos(4\theta)+\cos(2\theta)} \equiv \frac{\cancel{2}\sin\left(\frac{4\theta+2\theta}{2}\right)\cancel{\cos\left(\frac{4\theta-2\theta}{2}\right)}}{\cancel{2}\cancel{\cos\left(\frac{4\theta-2\theta}{2}\right)}\cos\left(\frac{4\theta+2\theta}{2}\right)}$$

$$\equiv \frac{\sin(3\theta)}{\cos(3\theta)} \equiv \tan(3\theta)$$

PROVE $\sin\theta[\sin(3\theta)+\sin(5\theta)] \equiv \cos\theta[\cos(3\theta)-\cos(5\theta)]$

$$\sin\alpha \pm \sin\beta \equiv 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\Leftrightarrow \sin\theta[\sin(3\theta)+\sin(5\theta)] = \sin\theta\left[2\sin\left(\frac{3\theta+5\theta}{2}\right)\cos\left(\frac{3\theta-5\theta}{2}\right)\right]$$

$$= 2\sin\theta\sin(4\theta)\cos(-\theta)$$

$$= 2\sin\theta\sin(4\theta)\cos\theta$$

and

$$\cos\alpha - \cos\beta \equiv -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\Leftrightarrow \cos\theta[\cos(3\theta)-\cos(5\theta)] = \cos\theta\left[-2\sin\left(\frac{3\theta-5\theta}{2}\right)\sin\left(\frac{3\theta+5\theta}{2}\right)\right]$$

$$= -2\cos\theta\sin(-\theta)\sin(4\theta)$$

$$= 2\sin\theta\sin(4\theta)\cos\theta$$

$$= \sin\theta[\sin(3\theta)+\sin(5\theta)]$$

PROVE $1+\cos(2\theta)+\cos(4\theta)+\cos(6\theta) \equiv 4\cos\theta\cos(2\theta)\cos(3\theta)$

$\sin\alpha \pm \sin\beta \equiv 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$ $\sin\alpha\sin\beta \equiv \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$

$\cos\alpha + \cos\beta \equiv 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ $\cos\alpha\cos\beta \equiv \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

$\cos\alpha - \cos\beta \equiv -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$ $\sin\alpha\cos\beta \equiv \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

$$4\cos\theta\cos(2\theta)\cos(3\theta) = 4\tfrac{1}{2}\left[\cos(2\theta-\theta)+\cos(2\theta+\theta)\right]\cos(3\theta)$$

$$= 2\left[\cos\theta\cos(3\theta) + \cos^2(3\theta)\right]$$

$$= 2\left(\tfrac{1}{2}\left[\cos(3\theta-\theta)+\cos(3\theta+\theta)\right] + \cos^2(3\theta)\right)$$

$$= \cos(2\theta)+\cos(4\theta)+2\cos^2(3\theta)$$

$$= \cos(2\theta)+\cos(4\theta)+\cos(6\theta)+1$$

$$= 1+\cos(2\theta)+\cos(4\theta)+\cos(6\theta)$$

PROVE $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

WHEN $\pi = \alpha+\beta+\gamma$

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha + \tan\beta + \tan(\pi-\alpha-\beta)$$

$$= \tan\alpha + \tan\beta - \tan(\alpha+\beta)$$

$$= \tan\alpha + \tan\beta - \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\tan\alpha(1-\tan\alpha\tan\beta)}{1-\tan\alpha\tan\beta} + \frac{\tan\beta(1-\tan\alpha\tan\beta)}{1-\tan\alpha\tan\beta} - \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$= -\frac{\tan^2\alpha\tan\beta + \tan\alpha\tan^2\beta}{1-\tan\alpha\tan\beta}$$

$$= -\tan\alpha\tan\beta\left(\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)$$

$$= -\tan\alpha\tan\beta\tan(\alpha+\beta)$$

$$= \tan\alpha\tan\beta\tan[-(\alpha+\beta)]$$

$$= \tan\alpha\tan\beta\tan\gamma$$