PRECALCULUS **Fall 2006** **EXAM #1** **SOLUTIONS**

#1 Which of the following are true?

a) $\sin(-25\pi) = \sin(113\pi)$ b) $\sin(\theta - 5\pi) = \sin(\theta - 11\pi)$

c) $\tan(2x) = \cot(x/2)$ d) $\cos(\theta + \pi) = \cos(\theta - \pi)$

e) $\sin(-\frac{24}{17}) = -\sin(\frac{24}{17})$

multiple of 2π

a) $\sin x$ is 2π periodic. $\sin(-25\pi) = \sin(-25\pi + 138\pi) = \sin(113\pi) \Rightarrow$ true

b) $\sin(\theta - 5\pi) = \sin(\theta - 5\pi - 6\pi) = \sin(\theta - 11\pi) \Rightarrow$ true

c) Not even close. Plug in a number for x. Try zero.

$\tan(2\cdot 0) = \tan(0) = 0$ but $\cot(0/2) = \cot(0) = $ undefined

remember $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$

d) $\cos(\theta + \pi) = \cos(\theta + \pi - 2\pi) = \cos(\theta - \pi) \Rightarrow$ true

e) $\sin x$ is an odd function, so $\sin(-x) = -\sin(x)$
so $\sin(-\frac{24}{17}) = -\sin(\frac{24}{17}) \Rightarrow$ true

#2 Which of the following are true?

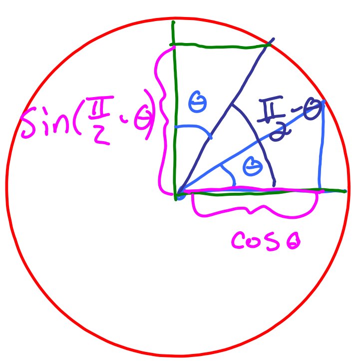
a) $\sin\theta$ is 2π periodic b) $\cos\theta = \sin(\frac{\pi}{2} - \theta)$

c) $\tan\theta$ is 2π periodic d) $\cot\theta$ has a period of $180°$

e) $\tan(3\theta - \pi/2)$ is $\frac{2\pi}{3}$ periodic

a) true

b) true



c) false. $\tan\theta$ is π periodic

d) true

e) $T = \frac{\pi}{\omega} = \frac{\pi}{3} \Rightarrow$ false

$\frac{\pi}{\omega}$ instead of $\frac{2\pi}{\omega}$

because $\tan x$ is π periodic

#3 Which of the following are true?

a) π is a rational number b) $s = r\theta$ where θ is in degrees

c) $\tan^2\theta = \sec^2\theta + 1$ is a pythagorean identity

d) I drop your 2 lowest exam grades e) $(\sin^2\theta - 1)\tan^2\theta = -\sin^2\theta$

a) False. π is irrational. $\pi = 3.141579....$ and continues forever without repetition. It can not be written as a fraction of an integer over an integer like rational numbers can.

b) False. $s = r\theta$ with θ in radians

c) False. $\tan^2\theta + 1 = \sec^2\theta \Rightarrow \tan^2\theta = \sec^2\theta - 1$

Remember $x^2 + y^2 = r^2 \Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$

$$\Rightarrow \cos^2\theta + \sin^2\theta = 1 \Rightarrow \begin{cases} \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow 1 + \tan^2\theta = \sec^2\theta \\ \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow \cot^2\theta + 1 = \csc^2\theta \end{cases}$$

d) False. I drop your lowest midterm exam grade.

e) LHS $= (\sin^2\theta - 1)\tan^2\theta = \sin^2\theta\tan^2\theta - \tan^2\theta$

Quick analysis: $\left.\begin{array}{l} \sin\theta \text{ is odd} \Rightarrow \sin^2\theta \text{ is even} \\ \tan\theta \text{ is odd} \Rightarrow \tan^2\theta \text{ is even} \end{array}\right\}$ because (odd)(odd) = even

example: $x \cdot x = x^2$

So, $\sin^2\theta\tan^2\theta - \tan^2\theta$ is even

even

Now lets look at the right hand side (RHS)

RHS $= -\sin^2\theta$ which is even.

So this quick analysis doesn't tell us this not an identity because both sides are even.

Lets try another quick analysis: plug in some easy numbers.

Try zero: LHS $(0) = (\sin^2(0) - 1)\tan^2(0) = (0-1)0 = 0$

RHS $(0) = -\sin^2(0) = 0$

So they match for this single number. This could still be an identity but remember, to be an identity, both sides must be equal for any number you plug in, not just one.

This problem should be easy enough to just "see" that this is an identity after doing so many HWK problems. continued on next page.

LHS $= (\sin^2\theta - 1)\tan^2\theta = -(1-\sin^2\theta)\tan^2\theta = -\cos^2\theta\tan^2\theta = -\sin^2\theta =$ RHS

You should be able to do this in your head by the time we finish Chapter 3. If this stuff bugs you, you need to spend some time "playing" with the identities.

#4 Which of the following are true?

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a) If the radius of a spinning bicycle tire is 5 cm then the tire's circumference is 25π cm.

b) The area of a sector of a circle is $\pi r\theta$

c) $\sin\theta = \frac{y}{r} = \frac{1}{\sec\theta}$ d) angular velocity $\omega = rt$ e) $\pi = 3.14\ldots$

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a) A tire has a radius of 5 cm whether it is spinning or not. The circumference of any circle is $C = 2\pi r$. So

$$C = 2\pi r = 2\pi(5\text{cm}) = 10\pi\text{cm} \neq 25\pi\text{cm} \Rightarrow \text{false}$$

b) This one is a dead give away. Not because I expected you to remember the "Area of a Sector" formula, or even derive it, for that matter (although you should be able to if you spent a few minutes looking at the derivation of the formula $s = r\theta$. You use the same trick) No, the big clue here is that an AREA should have units of AREA, like cm^2, in^2, m^2, any length squared.

$\pi r\theta$ only has r in it, not r^2. $\pi r\theta$ is a length. I mentioned this in class several times to help you keep the formulas for Areas, circumferences, volumes, whatever, organized in your head. Let your units guide you.

c) Almost but not quite. $\sin\theta = \frac{1}{y} = \frac{1}{\csc\theta} \neq \frac{1}{\sec\theta} \Rightarrow$ false.

Remember $\sin\theta = \frac{1}{\csc\theta}$ and $\cos\theta = \frac{1}{\sec\theta}$ the first letters of each function are mismatched.

d) Angular velocity. This is angle per time.

$\omega = rt$ has units of length·time, not angle per time. Again, a dead give away. $\omega = \underset{\text{angle}}{\theta} / \underset{\text{time}}{t}$ $\Rightarrow$ false

e) $\pi = 3.14\ldots$ Absolutely. Pi is a number. A number that goes on forever and does not repeat. You can not write the whole thing down except symbolically with "π". The ... is notation implying that something keeps going.

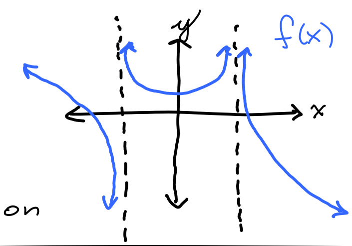
If I wrote $\pi = 3.14$, this is incorrect.

If I wrote $\pi \approx 3.14$ with a wavy equalls sign, this would be correct. $\approx$ means "approximately equal to".

#5 The symmetry of the function f(x) is

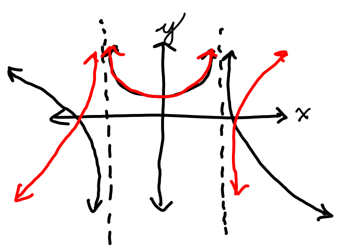
a) odd b) even c) both

d) neither e) this is not a function



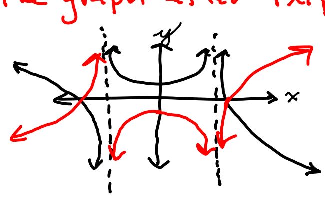
First off, a function can not be both even and odd.* Parts of a function can have even symmetry and parts can have odd symmetry, but overall, you get one or the other, or neither.

Check for even symmetry: flip it over the y-axis



It does not match everywhere.

Check for odd symmetry: Does flipping it over the y-axis give you the same graph as flipping it over the x-axis? We see what we get when flipping over the y-axis above. Lets look at the graph after flipping over the x-axis



This doesn't match the graph in red above so its not odd. Our original graph does pass the vertical line test so it is a function.

Not even, not odd, can't be both, is a function $\Rightarrow$ D neither.

#6 If the radius of a ferris wheel is 35 meters and one ride lasts 10 minutes, how fast does the ferris wheel have to spin for a rider to go 350π meters?

a) $70\pi \frac{\text{meters}}{\text{min}}$ b) $1 \frac{\text{rev}}{\text{min}}$ c) $70 \frac{\text{meters}}{\text{min}}$ d) $\frac{1}{2} \frac{\text{rev}}{\text{min}}$
e) $\pi \frac{\text{rad}}{\text{min}}$

A wheel with radius of 35 m.

A ride lasts 10 min.

How fast does it need to spin to go 350π meters?

Well, speed is length per time. We want to go a length 350πm. We have 10 min. to do so. This gives a speed of

$$V = \frac{350\pi \text{ m}}{10 \text{ min}} = \frac{35\pi \text{ m}}{\text{min}}.$$ This isn't an option.

Maybe some of the other options are equivalent. Lets do some dimensional analysis. Our options have units of $\frac{m}{min}$, $\frac{rev}{min}$, and $\frac{rad}{min}$

The m/min options aren't equal to what we got so we can ignore those

Lets try $\frac{rev}{min}$. 1 rev = go all the way around the wheel = $2\pi r$
$= 2\pi(35\text{m}) = 70\pi\text{m}$

So, $$35\pi \frac{m}{min} = 35\cancel{\pi} \frac{\cancel{m}}{min}\left(\frac{1 \text{ rev}}{70\cancel{\pi}\cancel{m}}\right) = \frac{1 \text{ rev}}{2 \text{ min}} \Rightarrow \text{D}$$

What about $\frac{rad}{min}$. Well 1 rev = 2π rad. So,

$$\frac{1 \text{ rev}}{2 \text{ min}} = \frac{1 \cancel{\text{rev}}}{\cancel{2} \text{ min}}\left(\frac{\cancel{2}\pi \text{ rad}}{1 \cancel{\text{rev}}}\right) = \pi \frac{\text{rad}}{\text{min}}$$ (half a rotation per minute)
$\Rightarrow$ E

#7 Which of the following are true?

a) $\pi^\circ = 180$ radians b) 2π rad $= 360^\circ$ c) $60^\circ 6' = 60.1^\circ$
d) $\pi \approx 3.14\ldots$ degrees e) $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

a) Nope. π rad $= 180^\circ$, not the other way around.

b) Yep. 2π rad $= 360^\circ$, both are a full rotation

c) Yep. If there are $60'$ per degree then $6'$ is $1/10$ of one degree

So, $60^\circ 6' = 60^\circ + 6' = 60^\circ + \left(\frac{1}{10}\right)^\circ = 60.1^\circ$

d) Nope. π is a number. You can assume units of radians if nothing is written if you can tell by context that it's refering to an angle. So if we do mean π rad, this is half the circle.

$3.14\ldots^\circ$ on, the other hand, is a very small portion of the circle, considering it takes 360° to go around the circle.

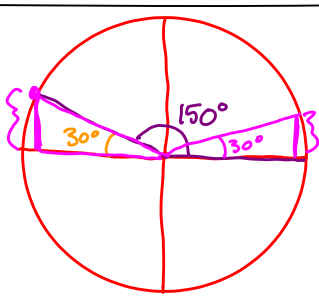
e) Yep. $\sin(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{1}{\sqrt{2}}$ I told you this would be on the test. Recognize equivalent fractions.

#8 $\sin(150^\circ) =$

a) $\sin(30^\circ)$ b) $\cos(60^\circ)$ c) $-\cos(300^\circ)$
d) $-\sin(330^\circ)$ e) $\tan(90^\circ)$

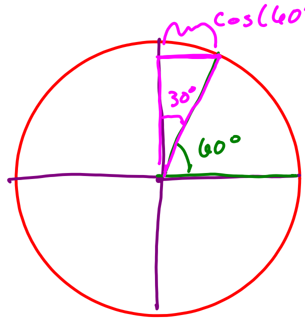
$\sin(150^\circ) = \sin(30^\circ) = 1/2$

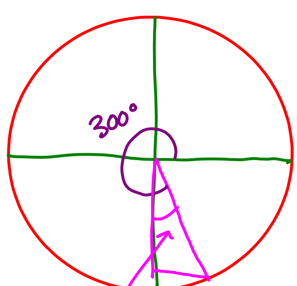
$\Rightarrow$ A



$\cos(60^\circ) = \sin(30^\circ) = \sin(150^\circ) = 1/2$

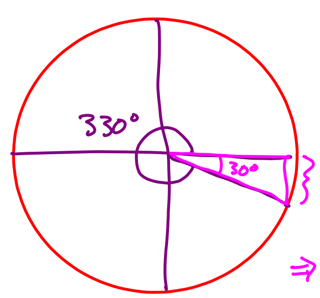
$\Rightarrow$ B





30° $\cos(300^\circ) = 1/2$

$\Rightarrow -\cos(300^\circ) = -1/2 \neq \frac{1}{2} = \sin(30^\circ) \Rightarrow$ c is false



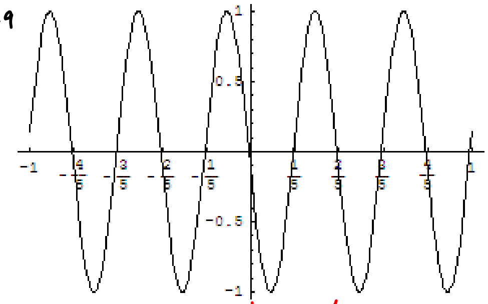
$\sin(330^\circ) = -1/2$

$\Rightarrow -\sin(330^\circ) = --1/2 = 1/2 = \sin(30^\circ)$

$\Rightarrow$ D

E. Not even close. $\tan(90^\circ)$ is undefined

#9



is a graph of which of the following functions?

a) $\sin\left(\frac{\pi x}{5}\right)$ b) $-\sin(5\pi x)$

c) $\sin(5\pi x - 3)$ d) $\sin(5\pi x - 5)$

e) $\sin(5\pi x - \pi)$

Has a period $T = 2/5 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{2/5} = 5\pi$

$\Rightarrow$ A is false.

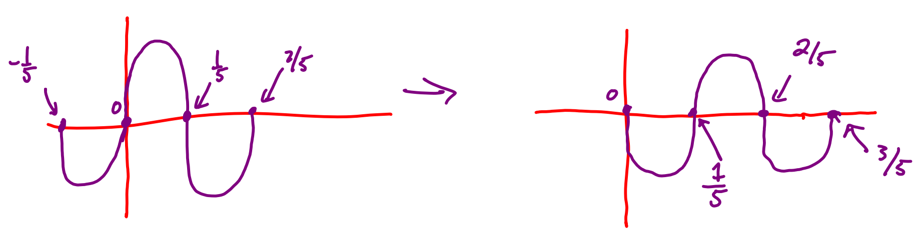
All are options involve $\sin(\cdots)$. This graph is a flipped version of $\sin(\cdots)$. Either over the x or the y axis. Or it could merely be shifted by half its period.

✓ B $-\sin(5\pi x)$ works.

✗ C $\sin(5\pi x - 3)$ is a shift $3/5\pi$ to the right. We have no π's labeled on our x-axis so this simply wouldn't line up. $\Rightarrow$ false

✗ D $\sin(5\pi x - 5)$ is a shift $5/5\pi = 1/\pi$ to the right. Again false.

✓ E $\sin(5\pi x - \pi)$ is a shift $\pi/5\pi = 1/5$ to the right. This works.



#10 Which of the following are completely true?

a) If you flip the graph of an odd function over the x-axis it will look the same as if you flipped it over the y-axis

b) If you flip the graph of an even function over the y-axis it will look the same as the original function.

c) letter (a) above is exemplified by the fact that $\sin(-x) = -\sin(x)$ where $\sin(-x)$ represents flipping $\sin(x)$ over the y-axis and $-\sin(x)$ represents flipping $\sin(x)$ over the x-axis

d) Although $-\sin(-x)$ flips the graph of $\sin(x)$ over the x-axis and then the y-axis (or vice versa), it can also be thought of as flipping $\sin(x)$ over the x-axis twice or the y-axis twice. This is one way to visualize the fact that $-\sin(-x) = \sin(x)$

e) $[\sin^2 x + \cos x]\tan x$ is an even function

Except for part E, this came directly from a sample exam.

(A) True. $-\sin(x) = \sin(-x)$

(B) True $\cos(-x) = \cos(x)$

(C) Yep.

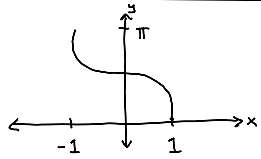
(D) yep again.

(E) $$[\sin^2 x + \cos x]\tan x = \underbrace{\underbrace{\underset{\text{odd}}{\sin x}\,\underset{\text{odd}}{\sin x}}_{\text{even}}\,\underset{\text{odd}}{\tan x}}_{\text{odd}} + \underbrace{\underset{\text{even}}{\cos x}\,\underset{\text{odd}}{\tan x}}_{\text{odd}}$$

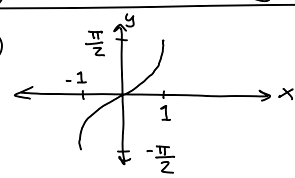
odd + odd is still odd. Multiplying changes odd an even symmetry.

$\Rightarrow$ false

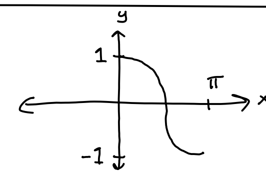
#11 Which of the following are completely true?

a) 

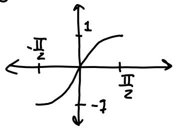
is a graph of $\sin^{-1}(x)$

b) 

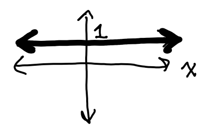
is a graph of $\cos^{-1}(x)$

c) 

is a graph of $\tan^{-1}(x)$

d) 

is a graph of $\sin^{-1}(x)$

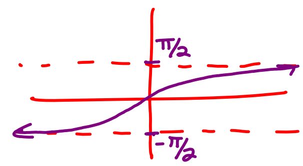
e) 

is a graph of $y = \sin^2 x + \cos^2 x$

a) Nope. $\sin^{-1}(x) \in [-\pi/2, \pi/2]$ not $[0,\pi]$

b) Nope. $\cos^{-1}(x) \in [0,\pi]$ not $[-\frac{\pi}{2}, \frac{\pi}{2}]$

c) Nope. $\tan^{-1}(x)$ looks like this



d) Nope. Need to switch x's and y's

e) Yep. $\sin^2 x + \cos^2 x = 1$ so the graph of $y = \sin^2 x + \cos^2 x$ is the same as the graph of $y = 1$

#12 Which of the following are completely true?

a) $\mathbb{R} \to \sin(x) \to [-1,1] \to \arcsin(x) \to [0,\pi]$

b) $\mathbb{R} \to \sin(x) \to (-1,1) \to \arcsin(x) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

c) $\mathbb{R} \to \cos(x) \to [-1,1] \to \arccos(x) \to [0,\pi]$

d) $[-1,1] \to \sin(x) \to [0,\pi] \to \arcsin(x) \to \mathbb{R}$

e) $\sin(\tan^{-1}(v)) = \frac{v}{\sqrt{1+v^2}}$

Again, from a sample exam, except for part (e).

A) Nope $\arcsin x \to [-\pi/2, \pi/2]$

B) Nope $\sin x \to [-1,1]$ with square brackets because we include -1, and 1

c) True

D) Nope. $\mathbb{R} \to \sin x \to [-1,1] \to \arcsin x \to [-\pi/2, \pi/2]$

E) True. We did this example in class

$\sin(\tan^{-1} v) = \sin\theta$ where $\theta = \tan^{-1} v$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\Rightarrow \tan\theta = v$ Two ways to solve from here.

① $\sin\theta = \sin\theta \frac{\cos\theta}{\cos\theta} = \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{v}{\sqrt{1+v^2}}$

② $\tan\theta = v = y/x$, let $y = v$ and $x = 1 \Rightarrow r = \pm\sqrt{x^2+y^2} = \pm\sqrt{1+v^2}$

$\Rightarrow \sin\theta = y/r = \frac{v}{\pm\sqrt{1+v^2}}$ to see that you want the positive version

realize than $\tan\theta = v$ so $v > 0 \Leftrightarrow \tan\theta > 0 \Leftrightarrow \sin\theta > 0$ since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

and $v < 0 \Leftrightarrow \tan\theta < 0 \Leftrightarrow \sin\theta < 0$ since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

#13 $\frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{\sin\theta} =$

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a) $2\csc\theta$ b) 0 c) $2\sin\theta$ d) $\tan\theta - \frac{1}{\cot\theta}$ e) $\sin^2\theta - (\cos^2\theta - 1)$

Quick symmetry analysis:

$$\frac{\overset{\text{odd}}{\sin\theta}}{\underbrace{1+\cos\theta}_{\text{even}}} \Big\} \ \frac{\text{odd}}{\text{even}} = \text{odd}$$

$$\frac{\overbrace{1-\cos\theta}^{}}{\underset{\text{odd}}{\sin\theta}} \Big\} \ \frac{\text{even}}{\text{odd}} = \text{odd}$$

$\Rightarrow \frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{\sin\theta}$ is odd

Do the "official" check if you like. Does $f(-x) = -f(x)$?

$$f(\theta) = \frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{\sin\theta}$$

$$\Rightarrow f(-\theta) = \frac{\sin(-\theta)}{1+\cos(-\theta)} - \frac{1-\cos(-\theta)}{\sin(-\theta)} = -\frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{-\sin\theta}$$

$$= -\left(\frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{\sin\theta}\right) = -f(\theta)$$

a) $2\csc\theta$ is odd, so maybe. Lets plug in some #'s to see if we can rule it out:

try zero: $\frac{\sin(0)}{1+\cos(0)} - \frac{1-\cos(0)}{\sin(0)} = 0 - \text{undefined}$ bad choice. pick another #

try $\frac{\pi}{2}$: $\frac{\sin(\pi/2)}{1+\cos(\pi/2)} - \frac{1-\cos(\pi/2)}{\sin(\pi/2)} = 0 - 0 = 0$ but

$2\csc(\pi/2) = \frac{2}{1} = 2 \neq 0 \Rightarrow$ can't be an identity

b) Zero is a tricky function. It is unique. It's function is the x-axis. It is the only function that has both odd and even symmetry (contradicting what I said previously about a function only being odd or even. $f(x) = 0$ is the only one capable of both) We need to check.

$\frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{\sin\theta} = 0 \Leftrightarrow \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ continued...

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}\left(\frac{1-\cos\theta}{1-\cos\theta}\right) = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{\sin\theta}{1+\cos\theta} - \frac{1-\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} - \frac{1-\cos\theta}{\sin\theta} = 0 \Rightarrow$$ B is true

This was done in class several times.

c) Now that we know this thing is identically zero we can easily rule out $2\sin\theta \Rightarrow$ false

d) $\tan\theta - \frac{1}{\cot\theta} = \tan\theta - \tan\theta = 0 \Rightarrow$ true

e) $\sin^2\theta - (\cos^2\theta - 1) = \sin^2\theta + (1-\cos^2\theta) = \sin^2\theta + \sin^2\theta = 2\sin^2\theta \neq 0$

$\Rightarrow$ false

#14 $\frac{1+\tan\theta}{1+\cot\theta} =$

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a) $\tan\theta$ b) $1-\sin\theta$ c) $\cot\theta$ d) 1 e) $\tan^2\theta$

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a) $\frac{1+\tan\theta}{1+\cot\theta} = \frac{\tan\theta(\cot\theta+1)}{1+\cot\theta} = \tan\theta$ This was done in class. true

b) Maybe. Let's plug in a number.

try $\pi/4$: $\frac{1+\tan(\pi/4)}{1+\cot(\pi/4)} = \frac{1+1}{1+1} = 1$ but

$1-\sin(\pi/4) = 1 - \frac{\sqrt{2}}{2} \neq 1 \Rightarrow$ false

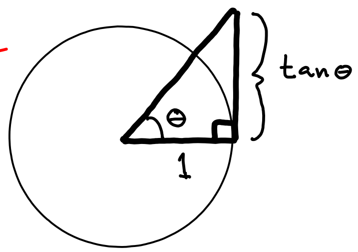
c) $\cot\theta \neq \tan\theta \Rightarrow$ false

d) $1 \neq \tan\theta \Rightarrow$ false

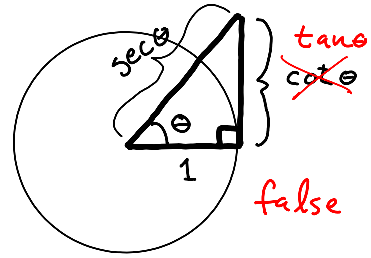
e) $\tan^2\theta \neq \tan\theta \Rightarrow$ false

#15 Which of the following unit circles is correctly labeled?

a) true

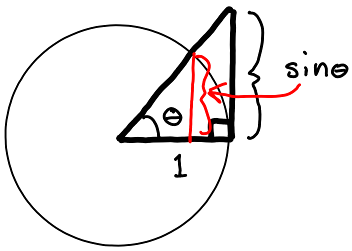


b)

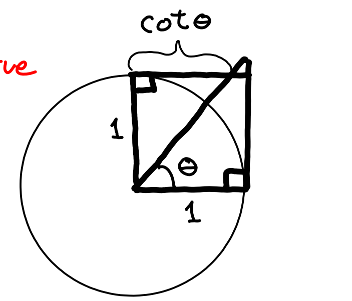


false

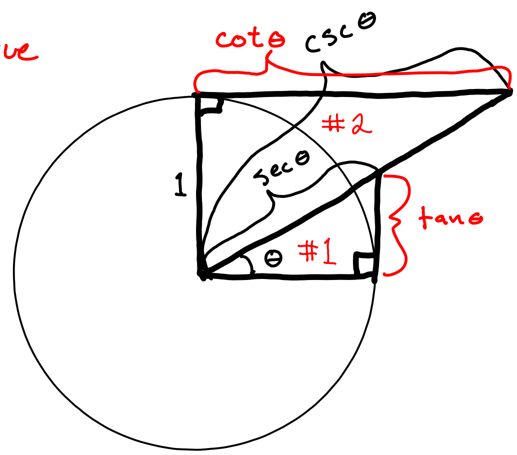
c) false



d) true

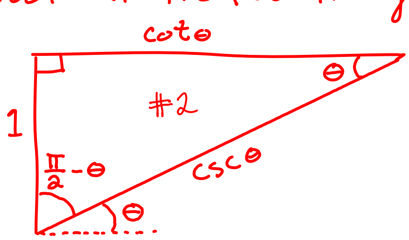


e) true

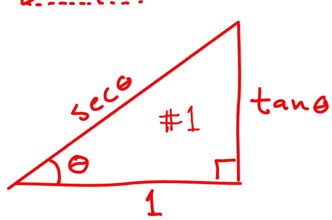


☆ Note: Remember your Pythagorean identities. There are 3 of them.

Look at the two triangles here.



remember $\cot^2\theta + 1 = \csc^2\theta$



remember $\tan^2\theta + 1 = \sec^2\theta$

These were all labeled on the diagram on the homepage of the course website. Take another look at it.