FALL 2006 PRECALCULUS EXAM #2 WITH SOLUTIONS

#1 Find <u>all</u> solutions to the equation $a\sin\theta - b\cos\theta = c$ where a, b and c are all positive real numbers.

a) $\dfrac{a}{\sqrt{a^2+b^2}} + 2n\pi$ b) $\sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi$ c) $\phi + \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi$ where ϕ is an angle such that $\cos\phi = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\phi = \dfrac{b}{\sqrt{a^2+b^2}}$

d) $\boxed{\begin{array}{l} \phi + \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi \\ \phi + \pi - \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi \\ \text{where } \alpha \text{ is an angle such that} \\ \cos\phi = \dfrac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\phi = \dfrac{b}{\sqrt{a^2+b^2}} \end{array}}$

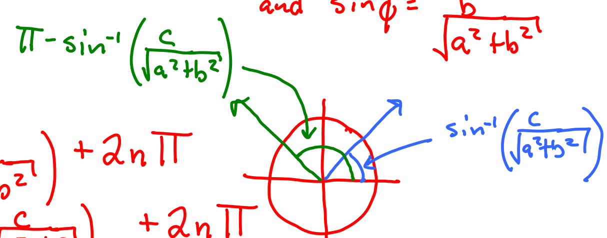
e) None of the above

$\cos\phi = \frac{x}{r}$ $\sin\phi = \frac{y}{r}$
let $x = a$, $y = b$
then $r = \sqrt{a^2+b^2}$
and $\cos\phi = \dfrac{a}{\sqrt{a^2+b^2}}$
and $\sin\phi = \dfrac{b}{\sqrt{a^2+b^2}}$

$$a\sin\theta - b\cos\theta = c \Leftrightarrow \frac{a}{\sqrt{a^2+b^2}}\sin\theta - \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

$$\Leftrightarrow \cos\phi\sin\theta - \sin\phi\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

$$\Leftrightarrow \sin\theta\cos\phi - \cos\theta\sin\phi = \frac{c}{\sqrt{a^2+b^2}}$$



$$\Leftrightarrow \sin(\theta - \phi) = \frac{c}{\sqrt{a^2+b^2}} \Leftrightarrow \begin{cases} \theta - \phi = \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi \\ \theta - \phi = \pi - \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi \end{cases}$$

$$\Leftrightarrow \boxed{\theta = \begin{cases} \phi + \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi \\ \phi + \pi - \sin^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right) + 2n\pi \end{cases} \quad \text{where } \phi \text{ is an angle such that } \cos\phi = \dfrac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\phi = \dfrac{b}{\sqrt{a^2+b^2}}}$$

#2 Find <u>all</u> solutions to the equation $2\sin(\theta^2 + 2\theta + 1) = 1$

a) $\boxed{\pm\sqrt{\pi/6 + 2n\pi} - 1,\ \pm\sqrt{5\pi/6 + 2n\pi} - 1}$ b) $\sin(\theta^2) = 1/2$, $\sin(2\theta) = 1/2$

c) $\pi/6 + 2n\pi$, $5\pi/6 + 2n\pi$ d) $(\pi/6 + 2n\pi)^2 + 2(\pi/6 + 2n\pi) + 1$, $(5\pi/6 + 2n\pi)^2 + 2(5\pi/6 + 2n\pi) + 1$

e) None of the above

$$2\sin(\theta^2 + 2\theta + 1) = 1 \Leftrightarrow \sin[(\theta+1)^2] = 1/2 \Leftrightarrow (\theta+1)^2 = \begin{cases} \pi/6 + 2n\pi \\ 5\pi/6 + 2n\pi \end{cases}$$

$$\Leftrightarrow \theta + 1 = \begin{cases} \pm\sqrt{\pi/6 + 2n\pi} \\ \pm\sqrt{5\pi/6 + 2n\pi} \end{cases} \Leftrightarrow \boxed{\theta = \begin{cases} \pm\sqrt{\pi/6 + 2n\pi} - 1 \\ \pm\sqrt{5\pi/6 + 2n\pi} - 1 \end{cases}}$$

#3 Find $\cos(3\theta)$ and $\sin(3\theta)$ in terms of 3rd degree polynomials in $\cos\theta$ and $\sin\theta$.

* A 3rd degree polynomial in x has the form $a_3x^3 + a_2x^2 + a_1x + a_0$ where a_3, a_2, a_1, a_0 are real numbers and are called the coefficients. Examples include $3x^3 + 6x^2 - x + 3$, $x^3 + 6x$, $9x^3 + x^2 - 6$, and $17x^3$. An nth degree polynomial has the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ or written in more compact notation $\sum_{i=0}^{n} a_i x^i$.

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a) $\cos(3\theta) = 3\cos^3\theta + 2\cos^2\theta + \cos\theta$
$\sin(3\theta) = \cos^3\theta - 2\cos^2\theta - 3\cos\theta$

b) $\cos(3\theta) = 6\cos^3\theta - 3\cos\theta$
$\sin(3\theta) = 3\sin^3\theta + 6\sin\theta$

c) $\boxed{\cos(3\theta) = 4\cos^3\theta - 3\cos\theta \quad \sin(3\theta) = -4\sin^3\theta + 3\sin\theta}$

d) $\cos(3\theta) = 3\cos^3\theta - 4\cos\theta$
$\sin(3\theta) = 4\sin^3\theta - 3\sin\theta$

e) None of the above

$\cos(3\theta) = \cos(\theta + 2\theta) = \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta)$ sum identity

$= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta)$ double angle identities

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$ pythagorean identity

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = \boxed{4\cos^3\theta - 3\cos\theta = \cos(3\theta)}$

$\sin(3\theta) = \sin(\theta + 2\theta) = \sin\theta\cos(2\theta) + \cos\theta\sin(2\theta)$ sum identity

$= \sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$ double angle identities

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$

$= -2\sin^3\theta + \sin\theta + 2\sin\theta(1 - \sin^2\theta)$ pythagorean identity

$= -2\sin^3\theta + \sin\theta + 2\sin\theta - 2\sin^3\theta = \boxed{-4\sin^3\theta + 3\sin\theta = \sin(3\theta)}$

#4 Find $\cos(6\theta)$ in terms of a 6th degree polynomial in $\cos\theta$ and $\sin\theta$.

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a) $\cos(6\theta) = 2^6\cos(6\theta) - 2^3\cos(3\theta) + 1$

b) $\boxed{\cos(6\theta) = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}$

c) $\cos(6\theta) = 48\cos^6\theta - 32\cos^4\theta + 18\cos^2\theta - 1$

d) $\cos(6\theta) = 6\cos^6\theta - 4\cos^2\theta + 6$

e) None of the above

$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ from previous problem

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$\cos(6\theta) = \cos(2\cdot 3\theta) = 2\cos^2(3\theta) - 1 = 2(4\cos^3\theta - 3\cos\theta)^2 - 1$

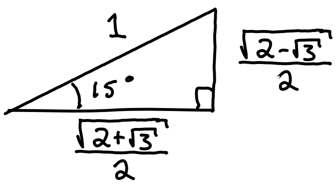
$= 2(16\cos^6\theta - 24\cos^4\theta + 9\cos^2\theta) - 1 = \boxed{32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \cos(6\theta)}$

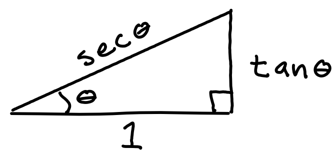
Note: Notice that the sum of the coefficients $32 - 48 + 18 - 1 = 1$ for $\cos(6\theta)$ and $4 - 3 = 1$ for $\cos(3\theta)$. Is this a pattern?

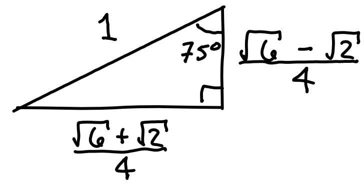
Consider, $\cos(n\theta) = \frac{1}{2}\left[\left(\cos\theta + \sqrt{\cos^2\theta - 1}\right)^n + \left(\cos\theta + \sqrt{\cos^2\theta - 1}\right)^{-n}\right]$

Is this an identity? It will be easy to tell once we master complex numbers.

#5 Which of the following triangles is labeled correctly?

a)


b)


c)


d) $\boxed{\text{All of the above}}$

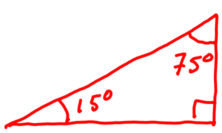
e) None of the above

$$\cos(15°) = \cos\left(\frac{30°}{2}\right) = \sqrt{\frac{1+\cos(30°)}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\sin(15°) = \sin\left(\frac{30°}{2}\right) = \sqrt{\frac{1-\cos(30°)}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\cos(15°) = \cos(45°-30°) = \cos(45°)\cos(30°) + \sin(45°)\sin(30°) = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sin(15°) = \sin(45°-30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°) = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$



$\cos(15°) = \sin(75°) = \frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$ and

$\sin(15°) = \cos(75°) = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

Remember your cofunction identities: $\sin(\theta) = \cos(\frac{\pi}{2}-\theta)$ and $\cos(\theta) = \sin(\frac{\pi}{2}-\theta)$
These hold true for the other trig functions too.

We can arrive at different but equivalent answers when using different methods.

We showed above using trig that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$. We'll show it again.

$$\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \Leftrightarrow \left[\frac{\sqrt{2+\sqrt{3}}}{2}\right]^2 = \left[\frac{\sqrt{6}+\sqrt{2}}{4}\right]^2 \longleftarrow$$

Squaring both sides is legit here because both sides were positive. This must be checked because $1^2 = (-1)^2$ but $1 \neq -1$

$$\Leftrightarrow \frac{2+\sqrt{3}}{4} = \frac{6+2\sqrt{6}\sqrt{2}+2}{16} \Leftrightarrow \frac{2+\sqrt{3}}{4} = \frac{8+2\sqrt{12}}{16}$$

$$\Leftrightarrow \frac{2+\sqrt{3}}{4} = \frac{8+2\sqrt{2\cdot 2\cdot 3}}{16} \Leftrightarrow \frac{2+\sqrt{3}}{4} = \frac{8+4\sqrt{3}}{16} = \frac{2+\sqrt{3}}{4} \checkmark$$

From the above knowledge we know triangles (a) and (c) are correctly labeled

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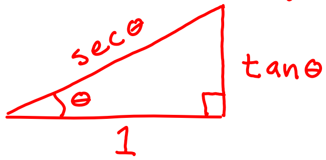
What about triangle (b)?



Remember your pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cot^2\theta + 1 = \csc^2\theta \quad \text{and} \quad \tan^2\theta + 1 = \sec^2\theta$$

The last equation tells you the triangle is labeled correctly. They are called pythagorean identities for a reason, they come from the Pythagorean Theorem and can be applied to right triangles.

We'll derive them again from the Pythagorean Theorem:

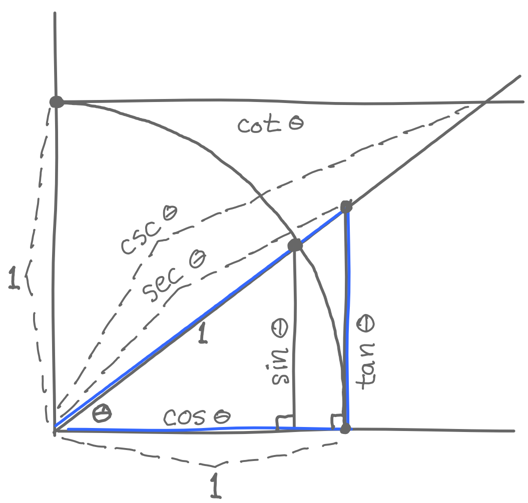
$$x^2 + y^2 = r^2 \Leftrightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} \Leftrightarrow \cos^2\theta + \sin^2\theta = 1 \quad \text{since } \sin\theta = \frac{y}{r} \text{ and } \cos\theta = \frac{x}{r}$$

And $\cos^2\theta + \sin^2\theta = 1 \Leftrightarrow \dfrac{\cos^2\theta}{\sin^2\theta} + \dfrac{\sin^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta} \Leftrightarrow \cot^2\theta + 1 = \csc^2\theta$

And $\cos^2\theta + \sin^2\theta = 1 \Leftrightarrow \dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \Leftrightarrow 1 + \tan^2\theta = \sec^2\theta$

So now you'll never be able to forget them...

Also, remember you're unit circle. Here's quadrant 1.



Thus all three triangles are labeled correctly and the answer is (d) All of the above

#6 Given that: $\sin\theta = $ Vader, $\cos\theta = $ Darth, $\tan\theta = $ Luke, $\csc\theta = $ Light, $\sec\theta = $ Saber, $\cot\theta = $ Skywalker, $\sin(2\theta) = $ Deathstar, $\cos(2\theta) = $ Anakin, $\tan(2\theta) = $ Obiwankenobi

Which of the following are false?

a) Luke Skywalker is Darth Vader

b) $\dfrac{1}{\text{Light}} - \dfrac{1}{\text{Saber}^2} + 5$ is $\text{Vader}^2 + \text{Vader} + 4$

c) Anakin is $\text{Darth}^2 - \text{Vader}^2$

d) None of the above (are false)

e) All of the above (are false)

a) Luke Skywalker $= \tan\theta \cot\theta = \tan\theta \cdot \dfrac{1}{\tan\theta} = 1$ but

Darth Vader $= \cos\theta \sin\theta \neq 1 \Rightarrow$ false

b) $\dfrac{1}{\text{Light}} - \dfrac{1}{\text{Saber}^2} + 5 = \dfrac{1}{\csc\theta} - \dfrac{1}{\sec^2\theta} + 5 = \sin\theta - \cos^2\theta + 5$

$= \sin\theta - (1 - \sin^2\theta) + 5 = \sin^2\theta + \sin\theta + 4$ and

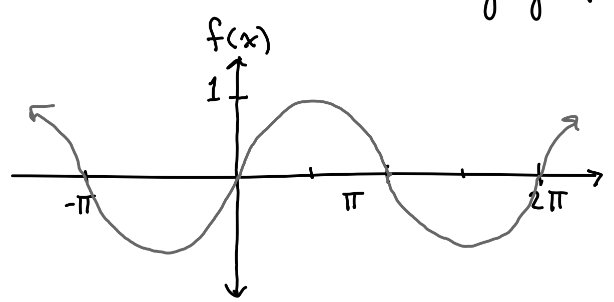
$\text{Vader}^2 + \text{Vader} + 4 = \sin^2\theta - \sin\theta + 4 \Rightarrow$ true

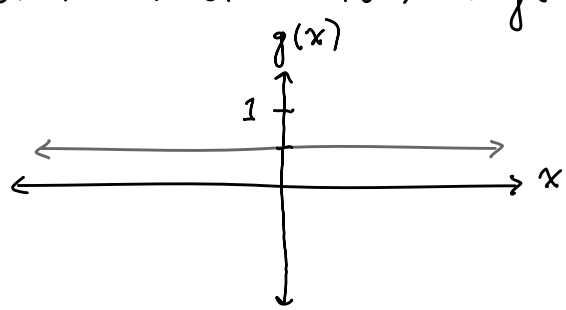
I actually worked out $\text{Vader}^2 + \text{Vader} + 4$ first and saw it was a polynomial in $\sin\theta$. So then I tried to reduce $\dfrac{1}{\text{Light}} - \dfrac{1}{\text{Saber}^2} + 5$ to the same.

c) Anakin $= \cos(2\theta)$ and

$\text{Darth}^2 - \text{Vader}^2 = \cos^2\theta - \sin^2\theta = \cos(2\theta) \Rightarrow$ true

#7 Consider the following graphs of two functions $f(x)$ and $g(x)$.





Which of the following are true?

(X) If $f(x) = \sin(x)$ then $g(x) = \frac{1}{2}[f^2(x) + \cos^2(x)]$

(Y) If $f(x) = \sin(x)$ then $g(x) = \frac{1}{2}[f^2(2x) + \cos^2(2x)]$

(Z) If $f(x) = \sin(x)$ then $g(2x) = 1$

a) $\boxed{\text{Only X and Y are true}}$ b) Only X is true c) Only X and Z are true
d) Only Y and Z are true e) X, Y, and Z are all true

By looking at the graph of $g(x)$, we can see that it is the graph of $g(x) = \frac{1}{2}$.
Assuming $f(x) = \sin(x)$,

(X) $g(x) = \frac{1}{2}[f^2(x) + \cos^2(x)] = \frac{1}{2}[\sin^2(x) + \cos^2(x)] = \frac{1}{2} \cdot 1 = \frac{1}{2} \Rightarrow$ X is true

(Y) $g(x) = \frac{1}{2}[f^2(2x) + \cos^2(2x)] = \frac{1}{2}[\sin^2(2x) + \cos^2(2x)] = \frac{1}{2} \cdot 1 = \frac{1}{2} \Rightarrow$ Y is true

Remember your identity $\sin^2\theta + \cos^2\theta = 1$. This is true for any angle, even when $\theta = 2x$. You can replace θ with anything as long as your replace it everywhere.

(Z) $g(2x) = g(x) = \frac{1}{2} \neq 1 \Rightarrow$ false $\Rightarrow$ Only X and Y are true $\Rightarrow$ Answer is (a)

Here, $g(x) = \frac{1}{2}$ is a constant function. It's $\frac{1}{2}$ everywhere no matter what you plug in for x. If, on the other hand, we had

$h(x) = 2x + 1$, then $h(2x) = 2(2x) + 1 = 4x + 1$

Anywhere there was an x in $h(x)$, we replaced it with $2x$ to find $h(2x)$. $g(x)$ had no x's in it, so it stays the same.

#8 Consider the values a, b, c, α, β, and γ where a, b, and c are the lengths of the sides of a triangle and α, β, and γ are the angles opposite sides, a, b, and c, respectively. Which of the following are true if $a = 1$ and $b = 2$.

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a) $\boxed{\text{If } \alpha = \pi/6 \text{ the only triangle that fits the data is a right triangle.}}$

b) If $0 < \alpha < \frac{\pi}{6}$ there are no triangles that fit the data.

c) If $\alpha = \beta$ then $c = \sqrt{5}$ d) All of the above

e) None of the above

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a) $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \Leftrightarrow \sin\beta = \frac{b}{a}\sin\alpha = 2\sin(\pi/6) = 1 \Rightarrow \beta = \pi/2 \Rightarrow$ true

b) $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \Leftrightarrow \sin\beta = \frac{b}{a}\sin\alpha = 2\sin\alpha$ which must be less than 1 to have a solution

$\sin\beta = 2\sin\alpha < 1 \Rightarrow \sin\alpha < 1/2 \Rightarrow \alpha < \pi/6 \Rightarrow$ There are no solutions if $\alpha > \pi/6 \Rightarrow$ false

c) $\alpha = \beta$, $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \Leftrightarrow \sin\alpha = \frac{\sin\alpha}{2} \Leftrightarrow \alpha = 0, \pi \Rightarrow$ Not a triangle $\Rightarrow$ false

#9 Find the exact value of $\tan[2\sin^{-1}(-3/5) + \tan^{-1}(3/4)]$

a) Undefined b) $3/5$ c) $3/4$ d) $\boxed{-3/4}$ e) None of the above

$$\tan[2\sin^{-1}(-3/5) - \tan^{-1}(3/4)] = \tan(2\alpha+\beta) = \frac{\tan(2\alpha)+\tan\beta}{1-\tan(2\alpha)\tan\beta}$$

where $\beta = \tan^{-1}(3/4) \Rightarrow \beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\tan\beta = 3/4$ and

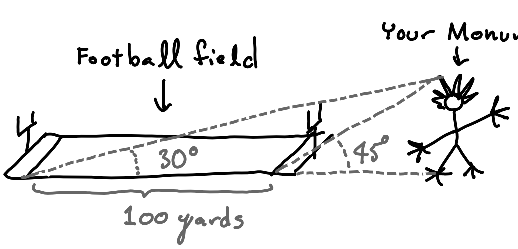
$\alpha = \sin^{-1}(-3/5) \Rightarrow \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin\alpha = -3/5 = y/r$, let $y = -3$, $r = 5$

$x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2 = 5^2 - (-3)^2 = 25 - 9 = 16 \Rightarrow x = \pm 4$, $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow x = 4$

$$\Rightarrow \tan\alpha = y/x = -3/4 \Rightarrow \tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2(-3/4)}{1-(3/4)^2} = \frac{-3/2}{1-9/16} = \frac{-3/2}{7/16} = -\frac{24}{7}$$

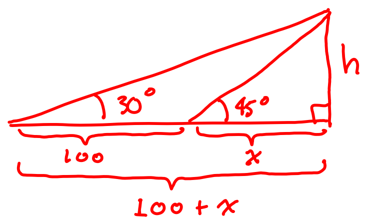
$$\Rightarrow \tan[2\sin^{-1}(-3/5) - \tan^{-1}(3/4)] = \frac{\tan(2\alpha)+\tan\beta}{1-\tan(2\alpha)\tan\beta} = \frac{-\frac{24}{7}+\frac{3}{4}}{1-\left(-\frac{24}{7}\right)\frac{3}{4}} = \frac{-\frac{96}{28}+\frac{21}{28}}{\frac{28}{28}+\frac{72}{28}} = \frac{-75}{100} = \boxed{-\frac{3}{4}}$$

#10 You have been awarded "Precalculus Student of the Universe" for the year. You have been given your own monument that stands in town square. You want to make sure your monument is taller than last year's "Precalculus Student of the Universe," so you need to measure the height of your monument. Near the monument is a football field. See the figure below. You use your trusty protractor to measure the angle of inclination to the top of the monument when standing at the zero yard line (end zone) at the closer end of the field find it to be about 45°. You do the same from the zero yardline at the other end of the field and find the angle of inclination to be about 30°. Luckily, the football field is oriented with the monument so that when you walk from the closer end zone to the other end zone, you walk directly away from the monument. Also, because your big sister is a professional football player, you know it's 100 yards from one end zone to the other. How tall is your monument in yards?



a) $\frac{100}{1+\sqrt{3}}$ b) $\boxed{\frac{100}{\sqrt{3}-1}}$ c) $\frac{100\sqrt{3}}{3}$

d) $\frac{100+\sqrt{3}}{2}$ e) None of the above



We have two right triangles. Each one gives a little part of the answer. We want to extract what info we can that relates to the height h from both triangles. Then combine this info to find the answer.

The bigger right triangle reveals $\tan(30°) = \frac{opp}{adj} = \frac{h}{100+x}$

The smaller right triangle reveals $\tan(45°) = \frac{opp}{adj} = \frac{h}{x}$

Two equations with two unknowns can be solved via substitution

$\tan(45°) = \frac{h}{x} \Rightarrow x = \frac{h}{\tan(45°)}$ substitute this into the other equation

$$\tan(30°) = \frac{h}{100+x} \Rightarrow h = (100+x)\tan(30°) = \left[100 + \frac{h}{\tan(45°)}\right]\tan(30°)$$

$$= 100\tan(30°) + h\frac{\tan(30°)}{\tan(45°)} = h$$

Now solve for h:

$$\Leftrightarrow h - h\frac{\tan(30°)}{\tan(45°)} = 100\tan(30°) \Leftrightarrow h\left[1 - \frac{\tan(30°)}{\tan(45°)}\right] = 100\tan(30°)$$

$$\Leftrightarrow h = \frac{100\tan(30°)}{1 - \frac{\tan(30°)}{\tan(45°)}} = \frac{100\,(1/\sqrt{3})}{1 - (1/\sqrt{3})/1} = \frac{100/\sqrt{3}}{\sqrt{3}/\sqrt{3} - 1/\sqrt{3}} = \frac{100/\sqrt{3}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{100}{\cancel{\sqrt{3}}}\left(\frac{\cancel{\sqrt{3}}}{\sqrt{3}-1}\right) = \boxed{\frac{100}{\sqrt{3}-1} = h}$$