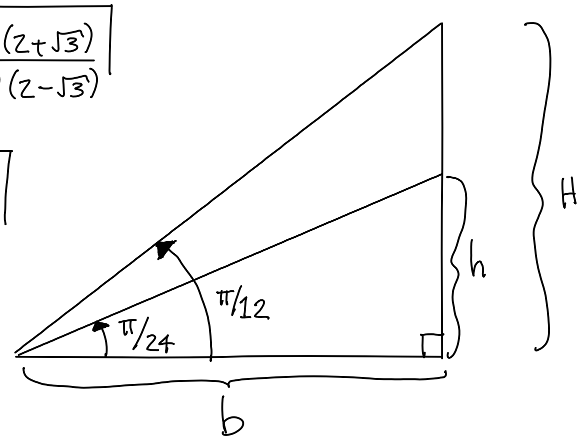
Solutions

#1) Using the figure and double/half angle identities, find h in terms of H.
Hint: Use half angle identities for $\sin\theta$ and $\cos\theta$ twice to find $\tan(\pi/24)$.



$\boxed{\text{A)}}$ $h=\frac{H}{2}\left(1-\frac{2-\sqrt{2+\sqrt{3}}}{2+\sqrt{2+\sqrt{3}}}\right)$ or $h=H\sqrt{\frac{(2-\sqrt{2+\sqrt{3}})(2+\sqrt{3})}{(2+\sqrt{2+\sqrt{3}})(2-\sqrt{3})}}$

B) $h=H\sqrt{\frac{2+\sqrt{2+\sqrt{3}}}{2-\sqrt{2+\sqrt{3}}}}$ or $h=H\sqrt{\frac{(\sqrt{2+\sqrt{3}})(2-\sqrt{3})}{(\sqrt{2+\sqrt{3}})(2+\sqrt{3})}}$

C) $h=H\left(2+\sqrt{2+\sqrt{3}}\right)/\left(2-\sqrt{2+\sqrt{3}}\right)$

or $h=H\left(2-\sqrt{2-\sqrt{3}}\right)/\left(2+\sqrt{2-\sqrt{3}}\right)$

D) All of the above E) None of the above

$$\left.\begin{array}{l}\tan(\pi/24)=h/b \Rightarrow h=b\tan(\pi/24)\\ \tan(\pi/12)=H/b \Rightarrow b=H/\tan(\pi/12)\end{array}\right\}\Rightarrow h=\frac{H\tan(\pi/24)}{\tan(\pi/12)}$$

$$\left.\begin{array}{l}\sin(\pi/12)=\sqrt{\frac{1-\cos(\pi/6)}{2}}=\frac{\sqrt{2-\sqrt{3}}}{2}\\ \cos(\pi/12)=\sqrt{\frac{1+\cos(\pi/6)}{2}}=\frac{\sqrt{2+\sqrt{3}}}{2}\end{array}\right\}\Rightarrow \tan(\pi/12)=\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$$

$$\left.\begin{array}{l}\sin(\pi/24)=\sqrt{\frac{1-\cos(\pi/12)}{2}}=\sqrt{\frac{1}{2}-\frac{1}{2}\cdot\frac{\sqrt{2+\sqrt{3}}}{2}}=\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}\\ \cos(\pi/24)=\sqrt{\frac{1+\cos(\pi/12)}{2}}=\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\frac{\sqrt{2+\sqrt{3}}}{2}}=\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}\end{array}\right\}\Rightarrow \tan(\pi/24)=\sqrt{\frac{2-\sqrt{2+\sqrt{3}}}{2+\sqrt{2+\sqrt{3}}}}$$

$$\tan\theta=\tan(\theta/2+\theta/2)=\frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}\Rightarrow\frac{\tan\theta}{\tan(\theta/2)}=\frac{2}{1-\tan^2(\theta/2)}\Rightarrow\frac{\tan(\theta/2)}{\tan\theta}=\frac{1-\tan^2(\theta/2)}{2}$$

$$\Rightarrow h=\frac{H\tan(\pi/24)}{\tan(\pi/12)}=H\left(\frac{1-\tan^2(\pi/24)}{2}\right)=\boxed{\frac{H}{2}\left(1-\frac{2-\sqrt{2+\sqrt{3}}}{2+\sqrt{2+\sqrt{3}}}\right)=h}$$

OR

$$h=\frac{H\tan(\pi/24)}{\tan(\pi/12)}=H\sqrt{\frac{2-\sqrt{2+\sqrt{3}}}{2+\sqrt{2+\sqrt{3}}}}\Bigg/\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}=\boxed{H\sqrt{\frac{(2-\sqrt{2+\sqrt{3}})(2+\sqrt{3})}{(2+\sqrt{2+\sqrt{3}})(2-\sqrt{3})}}=h}$$

#2) Solve for θ: $\sqrt{2\sin(\theta^2)} = \sqrt{\sqrt{3}}$

A) $\theta \in \left\{ \pm\sqrt{\pi/3} + 2k\pi,\ \pm\sqrt{2\pi/3} + 2k\pi\ ;\ k \in \mathbb{Z} \right\}$

B) $\theta \in \left\{ \pm\sqrt{\pi/3 + 2k\pi},\ \pm\sqrt{2\pi/3 + 2k\pi}\ ;\ k \in \mathbb{Z} \right\}$

C) $\theta \in \left\{ \sqrt{\pi/3 + 2k\pi},\ \sqrt{2\pi/3 + 2k\pi}\ ;\ k \in \mathbb{Z} \right\}$

D) $\theta \in \left\{ \pi/3 + 2k\pi,\ 2\pi/3 + 2k\pi\ ;\ k \in \mathbb{Z} \right\}$

E) None of the above

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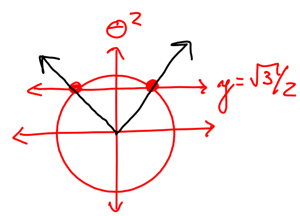
both sides of original equation are positive.

$\sqrt{x^2} = |x|$

$$\sqrt{2\sin(\theta^2)} = \sqrt{\sqrt{3}} \iff \left(\sqrt{2\sin(\theta^2)}\right)^2 = \left(\sqrt{\sqrt{3}}\right)^2 \iff |2\sin(\theta^2)| = |\sqrt{3}|$$

$$\iff \sin(\theta^2) = \sqrt{3}/2$$



$$\iff \theta^2 = \begin{cases} \pi/3 + 2k\pi \\ 2\pi/3 + 2k\pi \end{cases} \quad \text{where } k \in \mathbb{Z}$$

$$\iff \theta = \begin{cases} \pm\sqrt{\pi/3 + 2k\pi} \\ \pm\sqrt{2\pi/3 + 2k\pi} \end{cases} \quad \text{where } k \in \mathbb{Z}$$

$$\iff \boxed{\theta \in \left\{ \pm\sqrt{\pi/3 + 2k\pi},\ \pm\sqrt{2\pi/3 + 2k\pi}\ ;\ k \in \mathbb{Z} \right\}}$$

#3) Solve for θ: $\quad \sin(3\theta) = -\sin\theta$

A) $\theta \in \{2k\pi,\ \pi + 2k\pi;\ k \in \mathbb{Z}\}$

B) $\theta \in \{k\pi;\ k \in \mathbb{Z}\}$

$\boxed{\text{C)}}$ $\theta \in \{k\pi/2;\ k \in \mathbb{Z}\}$

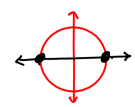
D) $\theta \in \{k\pi/2 + k\pi;\ k \in \mathbb{Z}\}$

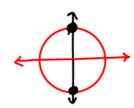
E) None of the above

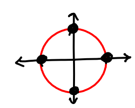
$$\begin{aligned}
\sin(3\theta) &= \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta \\
&= 4\sin\theta\cos^2\theta - \sin\theta
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sin(3\theta) = -\sin\theta \ &\Leftrightarrow\ 4\sin\theta\cos^2\theta - \sin\theta = -\sin\theta \\
&\Leftrightarrow\ 4\sin\theta\cos^2\theta = 0 \\
&\Leftrightarrow\ \sin\theta\cos^2\theta = 0 \\
&\Leftrightarrow\ \begin{cases} \sin\theta = 0 \ \Leftrightarrow\ \theta \in \{k\pi;\ k \in \mathbb{Z}\} \\ \cos\theta = 0 \ \Leftrightarrow\ \theta \in \{\pi/2 + k\pi;\ k \in \mathbb{Z}\} \end{cases} \\
&\Leftrightarrow\ \boxed{\theta \in \{k\pi/2;\ k \in \mathbb{Z}\}}
\end{aligned}$$







#4) Solve for θ: $\cos(3\theta) = -3\cos^3\theta$

A) $\theta \in \{ \pm\sqrt[3]{\pi/6 + 2K\pi} \,;\, K \in \mathbb{Z} \}$

B) $\theta \in \{ K\pi, (3K+1)\pi/2 \,;\, K \in \mathbb{Z} \}$

C) $\theta \in \{ K\pi/2 \,;\, K \in \mathbb{Z} \}$

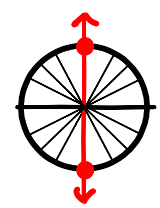
D) $\theta \in \{ (2K+1)\pi/2 \,;\, K \in \mathbb{Z} \}$

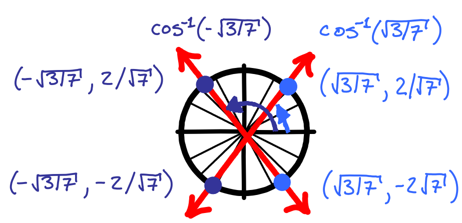
E) None of the above

$$\begin{aligned}
\cos(3\theta) &\equiv \cos(2\theta)\cos\theta - \sin\theta\sin(2\theta) \\
&\equiv (1-2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta \\
&= \cos\theta\,(1-4\sin^2\theta) \\
&= \cos\theta\,[1-4(1-\cos^2\theta)] \\
&= \cos\theta\,(-3+4\cos^2\theta) = -3\cos\theta + 4\cos^3\theta
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos(3\theta) = -3\cos^3\theta &\Leftrightarrow -3\cos\theta + 4\cos^3\theta = -3\cos^3\theta \\
&\Leftrightarrow -3\cos\theta + 4\cos^3\theta + 3\cos^3\theta = 0 \\
&\Leftrightarrow -3\cos\theta + 7\cos^3\theta = \cos\theta\,(-3+7\cos^2\theta) = 0 \\
&\Leftrightarrow \begin{cases} \cos\theta = 0 \Leftrightarrow \boxed{\theta \in \{ (2K+1)\pi/2 \,;\, K \in \mathbb{Z} \}} \\ -3+7\cos^2\theta = 0 \Leftrightarrow \cos^2\theta = 3/7 \Leftrightarrow \cos\theta = \pm\sqrt{3/7} \end{cases} \\
&\Leftrightarrow \boxed{\theta = \arccos(\pm\sqrt{3/7}) + \pi K}
\end{aligned}$$





Therefore,

$$\boxed{\cos(3\theta) = -3\cos^3\theta \Leftrightarrow \theta \in \{ (2K+1)\pi/2,\ \arccos(\pm\sqrt{3/7}) + \pi K \,;\, K \in \mathbb{Z} \}}$$

#5) Solve for θ: $3\tan(3\theta-\pi)=\sqrt{5}$

A) No real solutions.

B) $\theta = 3\tan^{-1}(\sqrt{5}/3)+\pi$

C) $\theta \in \{\pm\tan^{-1}(\sqrt{5}/3)+3k\pi ; k\in\mathbb{Z}\}$

D) $\theta = \tan^{-1}\left(\frac{\sqrt{5}}{3}\right)+\frac{k\pi}{3} ; k\in\mathbb{Z}\}$

$\boxed{\text{E)}}$ None of the above

$$3\tan(3\theta-\pi)=\sqrt{5} \Leftrightarrow \tan(3\theta-\pi)=\sqrt{5}/3$$
$$\Leftrightarrow \tan^{-1}(\tan(3\theta-\pi))=\tan^{-1}(\sqrt{5}/3)$$
$$\Leftrightarrow 3\theta-\pi = \tan^{-1}(\sqrt{5}/3)+k\pi ; k\in\mathbb{Z}$$
$$\Leftrightarrow 3\theta = \tan^{-1}(\sqrt{5}/3)+k\pi ; k\in\mathbb{Z}$$
$$\Leftrightarrow \boxed{\theta = \frac{1}{3}\left[\tan^{-1}\left(\frac{\sqrt{5}}{3}\right)+k\pi\right] ; k\in\mathbb{Z}}$$

#6) Solve for θ: $\sin^2\theta+\cos\theta=-3$

A) $\theta = (2k+1)\pi ; k\in\mathbb{Z}$

B) $\theta = 3\sin^{-1}(2/3)+\pi k ; k\in\mathbb{Z}$

C) $\theta \in \{\pm\cos^{-1}(1/3)+k\pi ; k\in\mathbb{Z}\}$

$\boxed{\text{D)}}$ No real solutions.

E) None of the above

$$\sin^2\theta+\cos\theta=-3 \Leftrightarrow 1-\cos^2\theta+\cos\theta+3=0 \Leftrightarrow \cos^2\theta-\cos\theta-4=0$$
$$\Leftrightarrow \cos\theta = \frac{1\pm\sqrt{1-4(1)(-4)}}{2(1)} = \frac{1\pm\sqrt{17}}{2}$$
$$\Leftrightarrow \begin{cases} \cos\theta = (1+\sqrt{17})/2 > 1 \Rightarrow \text{no solutions} \\ \cos\theta = (1-\sqrt{17})/2 < -1 \Rightarrow \text{no solutions} \end{cases}$$

$\boxed{\text{Thus, there are no real solutions.}}$

#7) Solve for θ: $\sin^2(2\theta)\cos^2(2\theta) = 1/4$

A) $\theta = (2K+1)\pi/2$; $K \in \mathbb{Z}$

B) $\theta = \pm(2K+1)\pi/4$; $K \in \mathbb{Z}$

$\boxed{\text{C)}}$ $\theta = (2K+1)\pi/8$; $K \in \mathbb{Z}$

D) No real solutions.

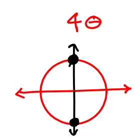
E) None of the above

$$\sin^2(2\theta)\cos^2(2\theta) = 1/4 \iff \sin(2\theta)\cos(2\theta) = \pm 1/2$$

$$\iff 2\sin(2\theta)\cos(2\theta) = \pm 1$$

$$\iff \sin(4\theta) = \pm 1$$

$$\iff 4\theta = \frac{\pi}{2} + K\pi = (2K+1)\frac{\pi}{2} \text{ ; } K \in \mathbb{Z}$$



$$\iff \boxed{\theta = (2K+1)\frac{\pi}{8} \text{ ; } K \in \mathbb{Z}}$$

#8) Solve for θ: $\sin^2(4\theta) - \cos^2(4\theta) = 0$

A) $\theta \in \{(2K+1)\pi/4, (2K+1)\pi/2 \text{ ; } K \in \mathbb{Z}\}$

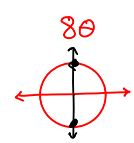
$\boxed{\text{B)}}$ $\theta = (2K+1)\pi/16$; $K \in \mathbb{Z}$

C) $\theta = K\pi/4$; $K \in \mathbb{Z}$

D) No real solutions.

E) None of the above

$$\sin^2(4\theta) - \cos^2(4\theta) = 0 \iff \cos(8\theta) = 0$$



$$\iff 8\theta = (2K+1)\pi/2 \text{ ; } K \in \mathbb{Z}$$

$$\iff \boxed{\theta = (2K+1)\pi/16 \text{ ; } K \in \mathbb{Z}}$$

#9) Solve for θ: $\quad \sin(3\theta) - \cos(3\theta) = 1/\sqrt{2}$

$\boxed{\text{A)}}$ $\theta \in \{ 5\pi/36 + 2k\pi/3 \,,\, 13\pi/36 + 2k\pi/3 \,;\, k \in \mathbb{Z} \}$

B) $\theta \in \{ \pm 5\pi/36 + 2k\pi/3 \,;\, k \in \mathbb{Z} \}$

C) $\theta \in \{ \pm 13\pi/36 + 2k\pi/3 \,;\, k \in \mathbb{Z} \}$

D) $\theta \in \{ 5\pi/36 + 2k\pi/3 \,,\, 13\pi/36 + 2k\pi/3 \,,\, 7\pi/36 + 2k\pi/3 \,;\, k \in \mathbb{Z} \}$

E) None of the above

$$\sin(3\theta) - \cos(3\theta) = 1/\sqrt{2} \iff \frac{\sqrt{2}}{2}\sin(3\theta) - \frac{\sqrt{2}}{2}\cos(3\theta) = \frac{\sqrt{2}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}$$

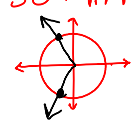
$$\iff \sin(\pi/4)\sin(3\theta) - \cos(\pi/4)\cos(3\theta) = 1/2$$

$$\iff -\left[\cos(\pi/4)\cos(3\theta) - \sin(\pi/4)\sin(3\theta)\right] = 1/2$$

$$\iff -\cos(3\theta + \pi/4) = 1/2$$

$$\iff \cos(3\theta + \pi/4) = -1/2$$

$3\theta + \pi/4$



$$\iff 3\theta + \pi/4 = \begin{cases} 2\pi/3 + 2k\pi \\ 4\pi/3 + 2k\pi \end{cases} \quad k \in \mathbb{Z}$$

$$\iff \theta = \begin{cases} \frac{1}{3}\left[\frac{2\pi}{3} - \frac{\pi}{4} + 2k\pi\right] = \frac{5\pi}{36} + \frac{2k\pi}{3} \\ \frac{1}{3}\left[\frac{4\pi}{3} - \frac{\pi}{4} + 2k\pi\right] = \frac{13\pi}{36} + \frac{2k\pi}{3} \end{cases} \quad k \in \mathbb{Z}$$

$$\iff \boxed{\theta \in \{ 5\pi/36 + 2k\pi/3 \,,\, 13\pi/36 + 2k\pi/3 \,;\, k \in \mathbb{Z} \}}$$

#10) Solve for θ: $\sqrt{3}\cos(\sqrt{3}\theta) + \sin(\sqrt{3}\theta) = \sqrt{3}$

A) $\theta \in \{ \pm\pi/\sqrt{3} + 2k\pi/\sqrt{3} \; ; \; k \in \mathbb{Z} \}$

B) $\theta \in \{ \pm 2\sqrt{3}k\pi/3 \; ; \; k \in \mathbb{Z} \}$

C) $\theta \in \{ \sqrt{3}\pi/9 + 2\sqrt{3}k\pi/3, \; 2\sqrt{3}k\pi/3 \; ; \; k \in \mathbb{Z} \}$

D) No real solutions

E) None of the above

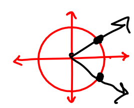
$$\sqrt{3}\cos(\sqrt{3}\theta) + \sin(\sqrt{3}\theta) = \sqrt{3} \iff \frac{\sqrt{3}}{2}\cos(\sqrt{3}\theta) + \frac{1}{2}\sin(\sqrt{3}\theta) = \frac{\sqrt{3}}{2}$$

$$\iff \cos(\pi/6)\cos(\sqrt{3}\theta) + \sin(\pi/6)\sin(\sqrt{3}\theta) = \sqrt{3}/2$$

$$\iff \cos(\sqrt{3}\theta - \pi/6) = \sqrt{3}/2$$

$$\iff \sqrt{3}\theta - \pi/6 = \pm \pi/6 + 2k\pi \; ; \; k \in \mathbb{Z}$$

$\sqrt{3}\theta - \pi/6$



$$\iff \sqrt{3}\theta = \begin{cases} \pi/3 + 2k\pi \\ 2k\pi \end{cases} ; \; k \in \mathbb{Z}$$

$$\iff \theta = \begin{cases} \sqrt{3}\pi/9 + 2\sqrt{3}k\pi/3 \\ 2\sqrt{3}k\pi/3 \end{cases} ; \; k \in \mathbb{Z}$$

$$\iff \boxed{\theta \in \{ \sqrt{3}\pi/9 + 2\sqrt{3}k\pi/3, \; 2\sqrt{3}k\pi/3 \; ; \; k \in \mathbb{Z} \}}$$