1. Find $2\cos(15^\circ)$ without using the double angle identity.

A) $\boxed{\dfrac{\sqrt{6}+\sqrt{2}}{2}}$ B) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ C) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ D) $\sqrt{2+\sqrt{3}}$ E) NONE OF THE ABOVE

$$2\cos(15^\circ) = 2\cos(45^\circ-30^\circ) = 2[\cos(45^\circ)\cos(30^\circ)+\sin(45^\circ)\sin(30^\circ)]$$

$$= 2\left[\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}\right] = 2\left[\frac{\sqrt{6}+\sqrt{2}}{4}\right] = \boxed{\frac{\sqrt{6}+\sqrt{2}}{2} = 2\cos(15^\circ)}$$

2. Find $\cos(7.5^\circ)$ and $\sin(\pi/24)$

A) $\cos(7.5^\circ) = \sqrt{\dfrac{2+\sqrt{6}+\sqrt{2}}{2}}$ and $\sin(\pi/24) = \sqrt{\dfrac{2-\sqrt{6}-\sqrt{2}}{2}}$

B) $\boxed{\cos(7.5^\circ) = \sqrt{\dfrac{4+\sqrt{6}+\sqrt{2}}{8}} \text{ and } \sin(\pi/24) = \sqrt{\dfrac{4-\sqrt{6}-\sqrt{2}}{8}}}$

C) $\cos(7.5^\circ) = \sqrt{\dfrac{2-\sqrt{3}}{4}}$ and $\sin(\pi/24) = \sqrt{\dfrac{2+\sqrt{3}}{4}}$

D) $\cos(7.5^\circ) = \dfrac{\sqrt{2+\sqrt{3}}}{2}$ and $\sin(\pi/24) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

E) NONE OF THE ABOVE

$$\cos(7.5^\circ) = \cos(15^\circ/2) = \sqrt{\frac{1+\cos(15^\circ)}{2}} = \sqrt{\frac{1}{2}\left[1+\frac{\sqrt{6}+\sqrt{2}}{4}\right]}$$

$$= \sqrt{\frac{1}{2}\left[\frac{4+\sqrt{6}+\sqrt{2}}{4}\right]} = \boxed{\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \cos(7.5^\circ)}$$

$$\sin(\pi/24) = \sin(7.5^\circ) = \sin(15^\circ/2) = \sqrt{\frac{1-\cos(15^\circ)}{2}} = \boxed{\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}} = \sin(\pi/24)}$$

3. If $\tan(4\theta) = 2/3$ and $4\theta \in [\pi, 3\pi/2)$, find $\cos(2\theta)$

A) $-\sqrt{\dfrac{1-\sqrt{3}}{2}}$ B) $\sqrt{\dfrac{1-3/\sqrt{13}}{2}}$

C) $\boxed{-\sqrt{\dfrac{1-3/\sqrt{13}}{2}}}$ D) $-\dfrac{\sqrt{\sqrt{3}+2}}{2}$

E) NONE OF THE ABOVE

$\tan(4\theta) = 2/3 \qquad \tan^2(4\theta) + 1 = \sec^2(4\theta) = \dfrac{1}{\cos^2(4\theta)}$ (Pythagorean Theorem)

$$\Rightarrow \cos(4\theta) = \pm\sqrt{\frac{1}{\tan^2(4\theta)+1}} = \pm\sqrt{\frac{1}{4/9+1}} = \pm\frac{3}{\sqrt{13}}$$

$$4\theta \in [\pi, 3\pi/2) \Rightarrow \cos(4\theta) < 0 \Rightarrow \cos(4\theta) = -3/\sqrt{13}$$

$\cos^2(\theta/2) = \dfrac{1+\cos\theta}{2} \Rightarrow \cos^2(2\theta) = \dfrac{1+\cos(4\theta)}{2}$ (Half Angle Identity)

$$\Rightarrow \cos(2\theta) = \pm\sqrt{\frac{1+\cos(4\theta)}{2}} = \pm\sqrt{\frac{1-3/\sqrt{13}}{2}}$$

$$4\theta \in [\pi, 3\pi/2) \Rightarrow \pi \le 4\theta < 3\pi/2 \Rightarrow \pi/2 \le 2\theta < 3\pi/4$$

$$\Rightarrow \cos(2\theta) \le 0 \Rightarrow \boxed{\cos(2\theta) = -\sqrt{\frac{1-3/\sqrt{13}}{2}}}$$

---

4. If $\cos\theta = x/r$ and $\sin\theta = y/r$ find $\tan(2\theta)$

A) $\dfrac{x^2-y^2}{r^2}$ $\quad$ $\boxed{\text{B) } \dfrac{2xy}{x^2-y^2}}$ $\quad$ C) $\dfrac{2xy}{r^2}$ $\quad$ D) $2\dfrac{y}{x}$ $\quad$ E) NONE OF THE ABOVE

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2(y/r)(x/r)}{(x/r)^2-(y/r)^2} = \boxed{\frac{2xy}{x^2-y^2} = \tan(2\theta)}$$

---

5. Solve for θ: $\sin(2\theta) + \sin(6\theta) = 0$

A) $\theta = \pi/4 + 2n\pi,\ 3\pi/4 + 2n\pi,\ n\pi,\ \pi/2 + 2n\pi$ $\quad$ B) $\theta = n\pi/2$

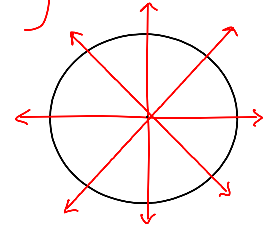
$\boxed{\text{C) } \theta = n\pi/4}$ $\quad$ D) $\theta = \pi/4 + n\pi/2$ $\quad$ E) NONE OF THE ABOVE

$$\sin(2\theta) + \sin(6\theta) = 0 \Leftrightarrow 2\cos\left(\frac{2\theta-6\theta}{2}\right)\sin\left(\frac{2\theta+6\theta}{2}\right) = 0$$

$$\Leftrightarrow 2\cos(-2\theta)\sin(4\theta) = 0 \Leftrightarrow 2\cos(2\theta)\sin(4\theta) = 0$$

$$\Leftrightarrow \left\{\begin{array}{l} \cos(2\theta)=0 \Leftrightarrow 2\theta = \pi/2 + 2n\pi \text{ and } 3\pi/2 + 2n\pi \Leftrightarrow 2\theta = \pi/2 + n\pi \\ \sin(2\theta) = 0 \Leftrightarrow 2\theta = 0 + 2n\pi \text{ and } \pi + 2n\pi \Leftrightarrow 2\theta = n\pi \end{array}\right\} \Leftrightarrow 2\theta = n\pi/2$$

$$\Leftrightarrow \boxed{\theta = n\pi/4 =} \{\ldots, 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4, \ldots\}$$



6. If $\pi = \alpha+\beta+\gamma$ then

$\boxed{\text{A) } \sin\gamma \equiv \sin(\alpha+\beta)}$ B) $\cos\gamma \equiv \sin(\alpha+\beta)$ C) $\tan\gamma \equiv \dfrac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)}$

D) ALL OF THE ABOVE E) NONE OF THE ABOVE

A) $\sin(\alpha+\beta) = \sin(\pi-\gamma) = \sin\left[2\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)\right] = 2\sin\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)\cos\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)$

$= 2\cos(\gamma/2)\sin(\gamma/2) = 2\sin(\gamma/2)\cos(\gamma/2) \underset{\uparrow}{=} \sin\gamma$

double angle identity

B) If $\alpha=\beta=0$ then $\pi = \alpha+\beta+\gamma \Rightarrow \gamma = \pi$. Since an identity is true for all values, it must be true for these, but

$\sin(\alpha+\beta) = \sin(0+0) = \sin(0) = 0$ and $\cos\gamma = \cos(\pi) = -1 \neq 0$

$\Rightarrow$ this cannot be an identity.

C) Try $\alpha=\beta=0$ and $\gamma=\pi$ again.

$\tan\gamma = \tan(\pi) = 0$ and $\dfrac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin(0)}{\cos(0)} = 0$

Maybe this is coincidence. Let's try $\beta=0$ and $\alpha = 3\pi/4$ and $\gamma = \pi/4$

$\tan\gamma = \tan(\pi/4) = 1$ but $\dfrac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin(3\pi/4)}{\cos(3\pi/4)} = \dfrac{\sqrt{2}/2}{-\sqrt{2}/2} = -1 \neq 1$

$\Rightarrow$ this cannot be an identity.

---

7. Solve for θ: $\cos^2(3\theta) - \sin^2(3\theta) = 0$

A) $\theta = \pi/4 + n\pi/2,\ \pi/12 + n\pi/2,\ 5\pi/12 + n\pi/2$

B) $\theta = 11\pi/12 + n\pi/6$

C) $\theta = \pi/4 + n\pi/6$

$\boxed{\text{D) ALL OF THE ABOVE}}$

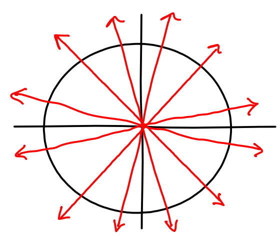
E) NONE OF THE ABOVE

$$\cos^2(3\theta) - \sin^2(3\theta) = 0 \iff [\cos(3\theta) - \sin(3\theta)][\cos(3\theta) + \sin(3\theta)] = 0$$

$$\iff \left.\begin{cases} \cos(3\theta) - \sin(3\theta) = 0 \iff \cos(3\theta) = \sin(3\theta) \iff 3\theta = \pi/4 + n\pi \\ \cos(3\theta) + \sin(3\theta) = 0 \iff \cos(3\theta) = -\sin(3\theta) \iff 3\theta = 3\pi/4 + n\pi \end{cases}\right\} \iff 3\theta = \pi/4 + n\pi/2$$

$$\iff \theta = 3\pi/12 + n\pi/6 = \pi/4 + n\pi/6 = \boxed{\pi/12 + n\pi/6 = \theta}$$

$$= \{\ldots, \pi/12, \pi/4, 5\pi/12, 3\pi/4, 11\pi/12, 13\pi/12, 5\pi/4, 17\pi/12, 19\pi/12, 7\pi/4, 23\pi/12, \ldots\}$$



Alternate Solution: Use Double-Angle Identity:

$$0 = \cos^2(3\theta) - \sin^2(3\theta) = \cos(6\theta) \Rightarrow 6\theta = \pi/2 + n\pi \Rightarrow \theta = \pi/12 + n\pi/6$$

---

8. Solve for θ: $\quad a\cos\theta - b\sin\theta = c \quad$ where $a, b,$ and c are positive constants and $a \neq 0$ or $b \neq 0$ and $c \leq \sqrt{a^2+b^2}$

A) $\theta = \alpha - \sin^{-1}\left(c/\sqrt{a^2+b^2}\right) + 2n\pi$ and $-\alpha + \cos^{-1}\left(c/\sqrt{a^2+b^2}\right) + 2n\pi$

where α satisfies $\cos\alpha = a/\sqrt{a^2+b^2}$ and $\sin\alpha = b/\sqrt{a^2+b^2}$

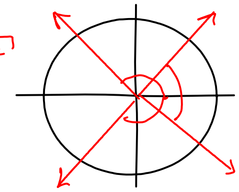
B) $\theta = -\alpha \pm \cos^{-1}\left(c/\sqrt{a^2+b^2}\right) + 2n\pi$ where $\cos\alpha = a/\sqrt{a^2+b^2}$ and $\sin\alpha = b/\sqrt{a^2+b^2}$

C) $\theta = \pm c/\sqrt{a^2+b^2}$ $\quad$ D) ALL OF THE ABOVE $\quad$ E) NONE OF THE ABOVE

Let $x = a$ and $y = b$, then $r = \sqrt{a^2+b^2}$ and there is some angle $\underbrace{\alpha \in [0, \pi/2]}_{\text{because } a \text{ and } b \text{ are positive.}}$

such that $\cos\alpha = \dfrac{x}{r} = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{y}{r} = \dfrac{b}{\sqrt{a^2+b^2}}$

$$a\cos\theta - b\sin\theta = c \iff \frac{a\cos\theta}{\sqrt{a^2+b^2}} - \frac{b\sin\theta}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}$$



$\pm\cos^{-1}\left[\dfrac{c}{\sqrt{a^2+b^2}}\right]$

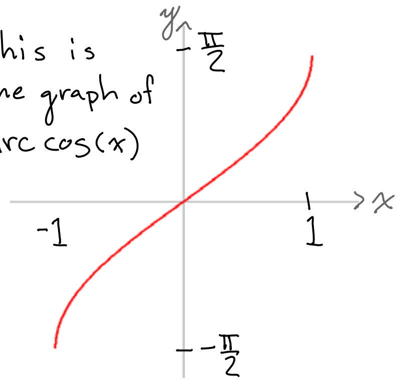
$$\iff \cos\alpha\cos\theta - \sin\alpha\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

$$\iff \cos(\alpha+\theta) = \frac{c}{\sqrt{a^2+b^2}} \implies \cos^{-1}[\cos(\alpha+\theta)] = \cos^{-1}\left[\frac{c}{\sqrt{a^2+b^2}}\right]$$

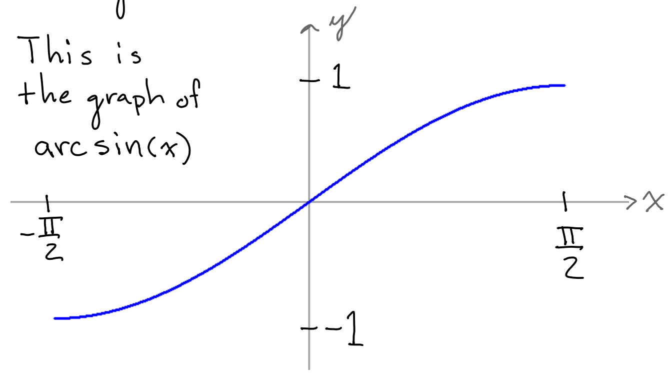
$$\implies \alpha + \theta = \pm\cos^{-1}\left[\frac{c}{\sqrt{a^2+b^2}}\right] + 2n\pi \implies \boxed{\theta = -\alpha \pm \cos^{-1}\left[\frac{c}{\sqrt{a^2+b^2}}\right] + 2n\pi}$$

9. Which of the following is completely true:

A) This is the graph of arc cos(x)



B) This is the graph of arc sin(x)

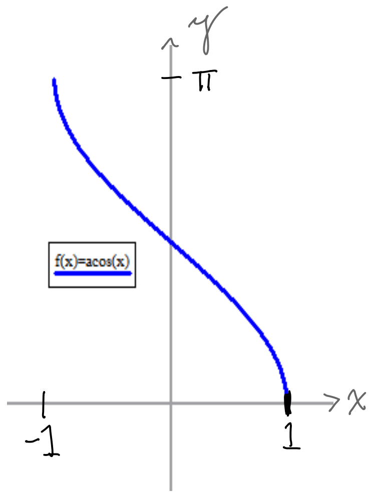


C) $\cos^{-1}[\cos(\cos^{-1}[\cos(5\pi/4)])] = \pi/4$

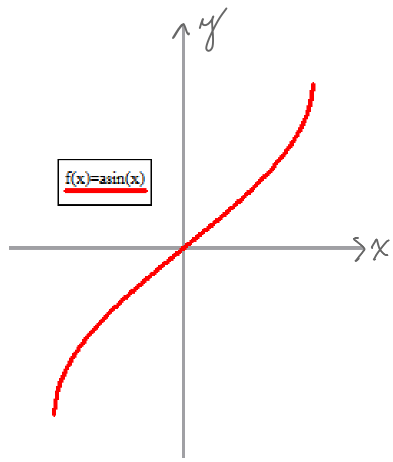
D) $\sin[\sin^{-1}(\sin[\sin^{-1}(\sqrt{3}/2)])] = \cos(\pi/3)$

E) NONE OF THE ABOVE

A)



B)



C) $\cos^{-1}[\cos(\cos^{-1}[\cos(5\pi/4)])] = \cos^{-1}[\cos(\cos^{-1}[-\sqrt{2}/2])]$

$= \cos^{-1}[\cos(3\pi/4)] = \cos^{-1}[-\sqrt{2}/2] = 3\pi/4 \neq \pi/4$

D) $\sin[\sin^{-1}(\sin[\sin^{-1}(\sqrt{3}/2)])] = \sin[\sin^{-1}(\sin[\pi/3])]$

$= \sin[\sin^{-1}(\sqrt{3}/2)] = \sin[\pi/3] = \sqrt{3}/2 \neq 1/2 = \cos(\pi/3)$

---

10. Which of the following correctly represents the domains and ranges of the specified functions and their inverses.

A) $\mathbb{R} \to \tan x \to \mathbb{R} \setminus \{\pi/2 + n\pi, n \in \mathbb{Z}\} \to \arctan x \to (-\pi/2, \pi/2)$

B) $\mathbb{R} \to \sin(x/2) \to [-1,1]$; $[-2,2] \to \arcsin(x/2) \to [-\pi,\pi]$

C) $\mathbb{R} \to \cos(2x) \to [-1,1]$; $[-1/2, 1/2] \to \arccos(2x) \to [0, \pi/2]$

$\boxed{\text{D) } \mathbb{R} \to \cos^2(x) \to [0,1] \text{ ; } [-1,1] \to \arccos^2(x) \to [0, \pi^2]}$

E) NONE OF THE ABOVE

A) False. $\mathbb{R} \setminus \{\pi/2 + n\pi, n \in \mathbb{Z}\} \to \tan x \to \mathbb{R} \to \arctan x \to (-\pi/2, \pi/2)$

B) False. $\mathbb{R} \to \sin(x/2) \to [-1,1]$; $[-2,2] \to \arcsin(x/2) \to [-\pi/2, \pi/2]$

Note: Consider extremes: $\arcsin(-2/2) = -\pi/2$, $\arcsin(2/2) = \pi/2$

C) False. $\mathbb{R} \to \cos(2x) \to [-1,1]$; $[-1/2, 1/2] \to \arccos(2x) \to [0,\pi]$

$\arccos(-2/2) = \pi$ and $\arccos(2/2) = 0$

D) True. $\mathbb{R} \to \cos(x) \to [-1,1] \to \arccos(x) \to [0,\pi]$

$\Rightarrow \mathbb{R} \to \underbrace{\cos^2(x)}_{\text{positive}} \to [0,1] \to \arccos^2(x) \to [0, \pi^2]$

positive: $\cos x \in [-1,1] \Rightarrow \cos^2 x \in [0,1]$

and $\arccos(x) \in [0,\pi] \Rightarrow \arccos^2(x) \in [0,\pi^2]$