1) Find $(1+i)^8(1+\sqrt{3}i)^3(2-2i)^4(\sqrt{3}-3i)^3(3-3i)^4$

A) $2^{15}3^9\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$ B) $2^{11}3^6\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

C) $2^{17}3^7$ D) $2^7 3^{11}$ E) $\boxed{\text{None of the above}}$

$$(1+i)^8(1+\sqrt{3}i)^3(2-2i)^4(\sqrt{3}-3i)^3(3-3i)^4 = z_1^8 z_2^3 z_3^4 z_4^3 z_5^4 \quad \text{where}$$

$$z_1 = 1+i = \sqrt{2}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right] = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right] \Rightarrow z_1^8 = 2^4\left[\cos(2\pi)+i\sin(2\pi)\right]$$

$$z_2 = 1+\sqrt{3}i = 2\left[\frac{1}{2}+\frac{\sqrt{3}}{2}i\right] = 2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right] \Rightarrow z_2^3 = 2^3\left[\cos(\pi)+i\sin(\pi)\right]$$

$$z_3 = 2-2i = 2\sqrt{2}\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right] = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right] \Rightarrow z_3^4 = 2^6\left[\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right]$$

$$z_4 = \sqrt{3}-3i = 2\sqrt{3}\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right] = 2\sqrt{3}\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right] \Rightarrow z_4^3 = 2^3 3^{3/2}\left[\cos(-\pi)+i\sin(-\pi)\right]$$

$$z_5 = 3-3i = 3\sqrt{2}\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right] = 3\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right] \Rightarrow z_5^4 = 2^2 3^4\left[\cos(-\pi)+i\sin(-\pi)\right]$$

$$\Rightarrow z_1^8 z_2^3 z_3^4 z_4^3 z_5^4 = 2^4\cdot 2^3\cdot 2^6\cdot 2^3\cdot 3^{3/2}\cdot 2^2\cdot 3^4\left[\cos\left(\pi-\frac{3\pi}{2}-\pi-\pi\right)+i\sin\left(\pi-\frac{3\pi}{2}-\pi-\pi\right)\right]$$

$$= 2^{18}\cdot 3^{11/2}\left[\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right] = \boxed{2^{18}3^{11/2}i}$$

2) Solve for z: $z^4 - 1 - i = 0$

$$\boxed{\text{A) } z_k = 2^{1/8}\left[\cos\left(\frac{\pi}{16}+\frac{\pi k}{2}\right)+i\sin\left(\frac{\pi}{16}+\frac{\pi k}{2}\right)\right] \quad k=0,\ldots,3}$$

B) $z_k = 2^{1/8}\left[\cos\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)\right]$ $k=0,\ldots,3$

C) $z_k = 2^{1/4}\left[\cos\left(\frac{\pi}{16}+\frac{\pi k}{4}\right)+i\sin\left(\frac{\pi}{16}+\frac{\pi k}{4}\right)\right]$ $k=1,\ldots,4$

D) $z_k = 2^{1/4}\left[\cos\left(\frac{\pi}{4}+\frac{\pi k}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi k}{4}\right)\right]$ $k=0,\ldots,3$

E) None of the above

$$z^4 - 1 - i = 0 \iff z^4 = 1 + i = \sqrt{2}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right] = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$$

$$z_k = \left(\sqrt{2}\right)^{1/4}\left[\cos\left(\frac{\pi/4}{4}+\frac{2\pi k}{4}\right)+i\sin\left(\frac{\pi/4}{4}+\frac{2\pi k}{4}\right)\right]$$

$$= \boxed{2^{1/8}\left[\cos\left(\frac{\pi}{16}+\frac{\pi k}{2}\right)+i\sin\left(\frac{\pi}{16}+\frac{\pi k}{2}\right)\right] = z_k \quad k=0,\ldots,3}$$

3) Which of the following choices of components of a triangle will fit 2 different triangles?

A) $a=2$, $b=3$, $\alpha=45°$ B) $a=3$, $b=2$, $\alpha=45°$

$\boxed{\text{C) } a=2,\ b=3,\ \alpha=30°}$ D) All of the above

E) None of the above

Hint: Use basic arithmetic to approximate and put upper and lower bounds on β_1 and β_2 when necessary.

A) $\sin\beta = \frac{b}{a}\sin\alpha = \frac{3}{2}\sin(45°) = \frac{3}{2}\cdot\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} > 1 \Rightarrow$ No solutions.

How are you supposed to know that $\frac{3\sqrt{2}}{4} > 1$?

Check: $\left(\frac{3\sqrt{2}}{4}\right)^2 = \frac{18}{16} > 1$ If $a>1$, then $a^2>1$ and if $a<1$, then $a^2<1$.

B) $\sin\beta = \frac{b}{a}\sin\alpha = \frac{2}{3}\sin(45°) = \frac{2}{3}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3} \Rightarrow \beta = \sin^{-1}\left(\frac{\sqrt{2}}{3}\right)$

$\sqrt{2} = 1.4\ldots < 1.5 = \frac{3}{2} \Rightarrow \frac{\sqrt{2}}{3} < \frac{3}{6} = \frac{1}{2} \Rightarrow \beta_1 = \sin^{-1}\left(\frac{\sqrt{2}}{3}\right) < \frac{\pi}{6} = 30°$

$\Rightarrow \beta_2 = 180° - \beta_1 > 150°$ but $\alpha = 45°$ and $\alpha + \beta_2 > 195° > 180°$

$\Rightarrow$ Only one triangle.

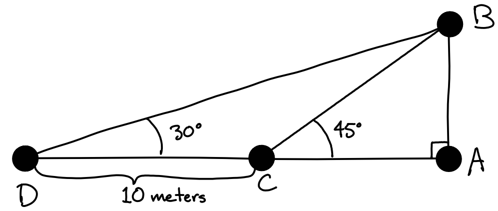
C) $\sin\beta = \frac{b}{a}\sin\alpha = \frac{3}{2}\sin(30°) = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4} < 1 \Rightarrow \beta_1 = \sin^{-1}\left(\frac{3}{4}\right)$

$\frac{\sqrt{2}}{2} < \frac{3}{4} < \frac{\sqrt{3}}{2} \Rightarrow 45° < \beta_1 < 60°$

$\beta_2 = 180° - \beta_1 \Rightarrow 120° < \beta_2 < 135°$

$\Rightarrow$ Look at max possibility of $135°$: $\beta_2 + \alpha = 165° < 180° \Rightarrow$ 2 possible triangles

4) Find the length of segment $\overline{AB}$ given $\overline{CD} = 10$ meters.



A) $\overline{AB} = 10 - \sqrt{3}$ meters

B) $\overline{AB} = 4\sqrt{3}$ meters

C) $\overline{AB} = \dfrac{10}{\sqrt{3}-1}$ meters

D) $\overline{AB} = \dfrac{5}{\sqrt{3}+1}$ meters

E) None of the above

$\tan(45°) = 1 = \dfrac{\overline{AB}}{x} \Rightarrow \overline{AB} = x$

$\tan(30°) = \dfrac{1}{\sqrt{3}} = \dfrac{\overline{AB}}{10+x} = \dfrac{\overline{AB}}{10+\overline{AB}} \Rightarrow 10 + \overline{AB} = \sqrt{3}\,\overline{AB} \Rightarrow \overline{AB}(1-\sqrt{3}) = -10 \Rightarrow \boxed{\overline{AB} = \dfrac{10}{\sqrt{3}-1}}$

5) Solve the triangle given $a=1$, $b=2$, and $c=3$

A) $\alpha = 32°$, $\beta = 64°$, $\gamma = 84°$
B) $\alpha = 16°$, $\beta = 32°$, $\gamma = 132°$
C) $\alpha = 32°$, $\beta = 16°$, $\gamma = 132°$
D) $\alpha = 26°$, $\beta = 39°$, $\gamma = 115°$
E) None of the above

$a^2 = b^2 + c^2 - 2bc\cos\alpha \Rightarrow \cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{12}{12} = 1 \Rightarrow \alpha = 0, 180°$

$\Rightarrow$ No solutions.

6) Solve for z: $z^2 + 2z = -2$

A) $z = -1 \pm 2i$
B) $z = -1+i, -1+2i$
C) $z = \pm 2i$
D) $z = -1 \pm i$
E) None of the above

$z^2 + 2z = -2 \Rightarrow z^2 - 2z + 2 = 0$

$\Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

7) Solve for z: $z^8 - 1 = 0$

A) $z = 1, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -1, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

B) $z = 1, \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}i, i, -\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}i, -1, -\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}i, -i, \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}i$

C) $z = 1, \frac{\sqrt{3}}{2} + \frac{1}{2}i, i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -1, -\frac{\sqrt{3}}{2} - \frac{1}{2}i, -i, \frac{\sqrt{3}}{2} - \frac{1}{2}i$

D) $z = 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$

E) None of the above

$z^8 - 1 = 0 \Rightarrow z^8 = 1 \Rightarrow 1$ is an obvious solution. Roots are equally spaced around a circle, in this case a circle of radius 1.

The angle between solutions is $\frac{2\pi}{8} = \frac{\pi}{4}$, thus the solutions can be written down.

Or you could use the root formula.

$$z_K = \sqrt[8]{1}\left[\cos\left(\frac{0}{8} + \frac{2\pi K}{8}\right) + i\sin\left(\frac{0}{8} + \frac{2\pi K}{8}\right)\right] \quad K = 0, \dots, 7$$

$z_0 = \cos(0) + i\sin(0) = 1$

$z_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$z_2 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

$z_3 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$z_4 = \cos(\pi) + i\sin(\pi) = -1$

$z_5 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

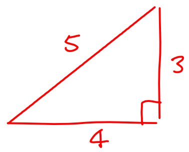
$z_6 = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$

$z_7 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

8) Find the area of the triangle with $a=4$, $b=3$, and $c=5$

A) 12 $\boxed{\text{B) } 6}$ C) $6\sqrt{2}$ D) $12\sqrt{3}$ E) None of the above

This should be recognized immediately as a 3,4,5 right triangle.



$$A = \frac{1}{2}(3)(4) = 6$$

Or use Heron's area formula: $s = \dfrac{a+b+c}{2} = 6$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{6(2)(3)(1)} = \sqrt{6\cdot 6} = \boxed{6 = A}$$

9) Find $\dfrac{(-1+i)^6}{(-1-i)^5}$

$\boxed{\text{A) } 1+i}$ B) $1-i$ C) $-1-i$ D) $-1+i$ E) None of the above

$$\frac{(-1+i)^6}{(-1-i)^5} = \frac{z_1^6}{z_2^5} \quad \text{where } z_1 = -1+i = \sqrt{2}\left[-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$\text{and } z_2 = -1-i = \sqrt{2}\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right] = \sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$

$$\Rightarrow \frac{z_1^6}{z_2^5} = \frac{(\sqrt{2})^6}{(\sqrt{2})^5}\left[\cos\left(6\cdot\frac{3\pi}{4} - 5\cdot\frac{5\pi}{4}\right) + i\sin\left(6\cdot\frac{3\pi}{4} - 5\cdot\frac{5\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right)\right] = \sqrt{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] = \boxed{1+i = \frac{(-1+i)^6}{(-1-i)^5}}$$

10) Find $\sin(41^\circ) - \cos(49^\circ)$

A) $2\cos(49^\circ)$ $\boxed{\text{B) } 0}$ C) $\dfrac{\sqrt{2}}{2}$ D) 1 E) None of the above

$$\sin(41^\circ) - \cos(49^\circ) = \cos(90^\circ - 41^\circ) - \cos(49^\circ) = \cos(49^\circ) - \cos(49^\circ) = \boxed{0 = \sin(41^\circ) - \cos(49^\circ)}$$