For problems 1-10 select the letter that has the correct answer.

Example: $\sin(\pi/6) =$ The correct answer would be A since $\sin(\pi/6) = 1/2$

A) $\pm 1/2$ B) ± 1 C) $\pm\sqrt{3}/2$ D) $\dfrac{\sqrt{6}\pm\sqrt{2}}{4}$ E) None of the above

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1) $\cos(\pi/12)\left[\sin^2(\pi/4)+\cos^2(\pi/4)\right] = \left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right)(1) = \dfrac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow D$

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2) $\cos(3\pi/4)\sin(3\pi/4)\left[\tan(3\pi/4)+\cot(3\pi/4)\right] = -\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{2}}{2}\left(-1-1\right) = 1 \Rightarrow B$

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3) $\cot(11\pi/12)\sin(11\pi/12) = \dfrac{\cos(11\pi/12)\cancel{\sin(11\pi/12)}}{\cancel{\sin(11\pi/12)}} = -\dfrac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow E$

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4) $\dfrac{\cos(7\pi/6)}{\cot(7\pi/6)} = \cancel{\cos(7\pi/6)}\,\dfrac{\sin(7\pi/6)}{\cancel{\cos(7\pi/6)}} = -\dfrac{1}{2} \Rightarrow A$

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5) $\sec(\pi/2-\pi/9)\sin(\pi/9) = \csc(\pi/9)\sin(\pi/9) = \dfrac{\sin(\pi/9)}{\sin(\pi/9)} = 1 \Rightarrow B$

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6) $\left[\cos^2(\pi/24)-\sin^2(\pi/24)\right]\left[\cos^2(\pi/24)+\sin^2(\pi/24)\right] = \cos(\pi/12)(1) = \dfrac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow D$

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7) $2\sin(30^\circ)\cos(15^\circ) - \sin(15^\circ) = 2\left(\dfrac{1}{2}\right)\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right) - \left(\dfrac{\sqrt{6}-\sqrt{2}}{4}\right) = \dfrac{\sqrt{2}}{2} \Rightarrow E$

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8) $\sin(\cos^{-1}(1/2)) = \sin(\pi/3) = \sqrt{3}/2 \Rightarrow C$

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9) $\sin(\cos^{-1}(\sin(\pi/3))) = \sin(\cos^{-1}(\sqrt{3}/2)) = \sin(\pi/6) = 1/2 \Rightarrow A$

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10) $\cot\left(\cos^{-1}\left[\sin(\tan^{-1}(1))\right]\right) = \cot\left(\cos^{-1}\left[\sin(\pi/4)\right]\right) = \cot(\cos^{-1}(\sqrt{2}/2))$

$= \cot(\pi/4) = 1 \Rightarrow B$

11) Which of the following are completely true?

A) $\cos(8x) = \cos^2(4x) - \sin^2(4x)$

B) $\cos(4x) = 1 - 2\sin^2(2x)$

C) $1 = 2\cos^2(3x) - \cos(6x)$

D) All of the above

E) None of the above

Double Angle Identity

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$\Rightarrow 1 = 2\cos^2\theta - \cos(2\theta)$$

if $\theta = 4x$ then $\cos(8x) = \cos^2(4x) - \sin^2(4x)$

if $\theta = 2x$ then $\cos(4x) = 1 - 2\sin^2(2x)$

if $\theta = 3x$ then $1 = 2\cos^2(3x) - \cos(6x)$

12) Which of the following are completely true if $x+y+z=\pi$?

A) $\boxed{\sin(x+y+z) = \sin(x+y)\cos z + \cos[-(x+y)]\sin z}$

B) $\cos(x+y-z) = \cos x\cos y\cos z - \sin x\sin y\sin z + \sin x\cos y\sin z + \cos x\sin y\sin z$

C) $\boxed{\sin x\cos(y+z) + \cos x\sin(y+z) = 0}$

D) All of the above

E) None of the above

A) $\sin(x+y+z) = \sin[(x+y)+z] = \sin(x+y)\cos z + \cos(x+y)\sin z$ (Sum Identity)

$= \sin(x+y)\cos z + \cos[-(x+y)]\sin z$ ✓ (Cosine is even)

B) $\cos(x+y-z) = \cos[(x+y)-z] = \cos(x+y)\cos z + \sin(x+y)\sin z$ (Sum Identity)

$= [\cos x\cos y - \sin x\sin y]\cos z + [\sin x\cos y + \cos x\sin y]\sin z$ (Sum Identity)

$= \cos x\cos y\cos z - \sin x\sin y\boxed{\cos z} + \sin x\cos y\sin z + \cos x\sin y\sin z$

$\neq \cos x\cos y\cos z - \sin x\sin y\boxed{\sin z} + \sin x\cos y\sin z + \cos x\sin y\sin z$ ✗

C) $\sin x\cos(y+z) + \cos x\sin(y+z) = \sin(x+y+z) = \sin(\pi) = 0$ ✓

I'll accept either A or C as correct. Pointing out this error (having 2 correct solutions) was worth extra credit.

13) If $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ which of the following are completely true?

A) $\omega^2 = \bar{\omega}$

B) $\bar{\omega}^2 = \omega$

C) $\boxed{\omega\bar{\omega} = 1}$

D) All of the above

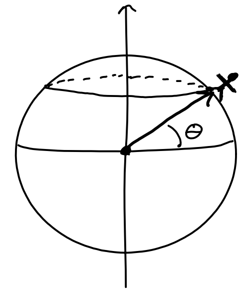
E) None of the above

A) $\omega^2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - \frac{3}{4} + \frac{\sqrt{3}}{2}i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \neq \frac{1}{2} - \frac{\sqrt{3}}{2}i = \bar{\omega}$

B) $\bar{\omega}^2 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}i = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \neq \frac{1}{2} - \frac{\sqrt{3}}{2}i = \bar{\omega}$

C) $\omega\bar{\omega} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{1}{4} + \frac{3}{4} = 1$ ✓

14) R_E is the Earth's radius. It takes 24 hours for the Earth to complete a single rotation about its axis. If you are standing on the surface of the spinning Earth at a latitude θ, what is your angular velocity, ω, and your linear velocity v?



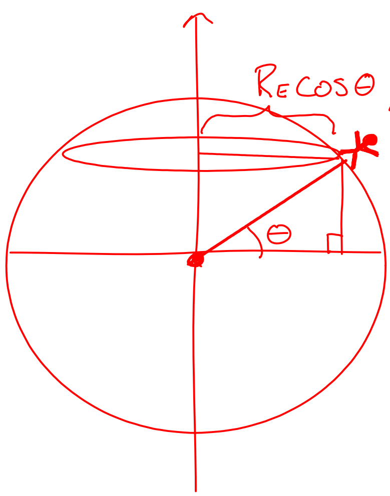
A) $v = \frac{\pi}{12\text{hours}}$, $\omega = \frac{\pi R_E}{12\text{ hours}}$

B) $v = \frac{\pi R_E \cos\theta}{12\text{ hours}}$, $\omega = \frac{\pi}{12\text{hours}}$

C) $v = \frac{\pi R_E}{12\text{ hours}}$, $\omega = \frac{\pi}{12\text{hours}}$

D) $v = \frac{\pi R_E \sin\theta}{12\text{ hours}}$, $\omega = \frac{\pi}{12\text{hours}}$

E) None of the above



standing here, you are going around in a circle of radius $R_E\cos\theta$ and has a circumference $2\pi R_E \cos\theta$.

You make 1 revolution every 24 hours

$$\Rightarrow v = \frac{2\pi R_E\cos\theta}{24\text{hours}} = \boxed{\frac{\pi R_E \cos\theta}{12\text{ hours}} = v}$$

No matter where you stand on the Earth,

$$\omega = \frac{\text{angle}}{\text{time}} = \frac{2\pi}{24\text{hours}} = \boxed{\frac{\pi}{12\text{ hours}} = \omega}$$

15) Which of the following are completely true?

A) $\cos(1^\circ) - \cos(2^\circ) + \cos(3^\circ) - \cos(4^\circ) + \cdots + \cos(359^\circ)$

$= -\sin(1^\circ) + \sin(2^\circ) - \sin(3^\circ) + \sin(4^\circ) + \cdots - \sin(359^\circ)$

B) $\cos^2(1^\circ) + \cos^2(2^\circ) + \cdots + \cos^2(359^\circ) = 1$

C) $\boxed{\tan(1^\circ) + \tan(2^\circ) + \cdots + \tan(89^\circ) + \tan(91^\circ) + \cdots + \tan(179^\circ) = \tan(181^\circ) + \cdots + \tan(269^\circ) + \tan(271^\circ) + \cdots + \tan(359^\circ)}$

D) All of the above

E) None of the above

A) $\cos(1^\circ) - \cos(2^\circ) + \cos(3^\circ) - \cos(4^\circ) + \cdots + \cos(359^\circ)$

$= \cos(1^\circ) + \cos(179^\circ) + \cos(3^\circ) + \cos(177^\circ) + \cdots + \cos(79^\circ) + \cos(91^\circ)$

$- [\cos(2^\circ) + \cos(178^\circ) + \cos(4^\circ) + \cos(176^\circ) + \cdots \cos(88^\circ) + \cos(92^\circ)] + \overset{0}{\cancel{\cos(90^\circ)}} + \overset{-1}{\cancel{\cos(180^\circ)}}$

$= \cos(1^\circ) - \cos(1^\circ) + \cos(3^\circ) - \cos(3^\circ) + \cdots + \cos(79^\circ) - \cos(79^\circ)$

$- [\cos(2^\circ) - \cos(2^\circ) + \cos(4^\circ) - \cos(4^\circ) + \cdots + \cos(88^\circ) - \cos(88^\circ)] - 1$

$= 0 - 0 - 1 = -1$ but

$-\sin(1^\circ) + \sin(2^\circ) - \sin(3^\circ) + \sin(4^\circ) + \cdots - \sin(359^\circ)$

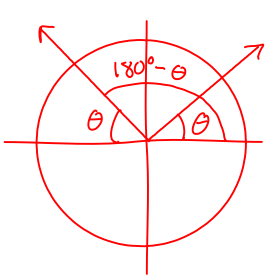
$= -[\sin(1^\circ) + \sin(359^\circ) + \sin(3^\circ) + \sin(357^\circ) + \cdots + \sin(179^\circ) + \sin(181^\circ)]$

$+ \sin(2^\circ) + \sin(358^\circ) + \sin(4^\circ) + \sin(356^\circ) + \cdots + \sin(178^\circ) + \sin(182^\circ) + \overset{0}{\cancel{\sin(180^\circ)}}$

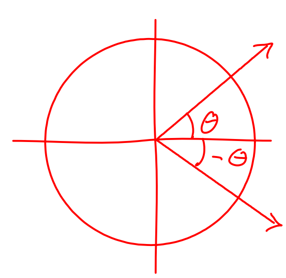
$= -[\sin(1^\circ) - \sin(1^\circ) + \sin(3^\circ) - \sin(3^\circ) + \cdots + \sin(179^\circ) - \sin(179^\circ)]$

$+ \sin(2^\circ) - \sin(2^\circ) + \sin(4^\circ) - \sin(4^\circ) + \cdots + \sin(178^\circ) - \sin(178^\circ) = 0 \neq -1$

This can be visualized simply by looking at the unit circle:



$\cos\theta = -\cos(180^\circ - \theta)$



$\sin\theta = -\sin(-\theta)$

B) $\cos^2(1^\circ) + \cos^2(2^\circ) + \cdots + \cos^2(359^\circ) = \overbrace{\cos^2(1^\circ) + \cos^2(2^\circ) + \cdots}^{>0} + \overset{1}{\cancel{\cos(180^\circ)^2}} + \overbrace{\cdots + \cos^2(359^\circ)}^{>0} > 1 \Rightarrow \neq 1$

C) $\tan(1^\circ)+\tan(2^\circ)+\cdots+\tan(89^\circ)+\tan(91^\circ)+\cdots+\tan(179^\circ)$

$= \tan(1^\circ)+\tan(179^\circ)+\tan(2^\circ)+\tan(178^\circ)+\cdots+\tan(89^\circ)+\tan(91^\circ)$

$= \tan(1^\circ)-\tan(1^\circ)+\tan(2^\circ)-\tan(2^\circ)+\cdots+\tan(89^\circ)-\tan(89^\circ)=0$ and

$\tan(181^\circ)+\cdots+\tan(269^\circ)+\tan(271^\circ)+\cdots+\tan(359^\circ)$

$= \tan(181^\circ)+\tan(359^\circ)+\tan(182^\circ)+\tan(358^\circ)+\cdots+\tan(269^\circ)+\tan(271^\circ)$

$= \tan(181^\circ)-\tan(181^\circ)+\tan(182^\circ)-\tan(182^\circ)+\cdots+\tan(269^\circ)-\tan(269^\circ)=0$ ✓

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16) $\sqrt{3}\,(1+i)^8(-1+\sqrt{3}i)^3(-1-\sqrt{3}i)^6(-1+i)^4=$

A) $2^{17}3^{1/2}$ B) $-2^{17}3^{1/2}$ C) $2^{17}3^{1/2}i$ D) $-2^{17}3^{1/2}i$ $\boxed{\text{E) None of the above}}$

$$\sqrt{3}\,(1+i)^8(-1+\sqrt{3}i)^3(-1+\sqrt{3}i)^6(-1+i)^4 = \sqrt{3}\,z_1^8 z_2^3 z_3^6 z_4^4 \quad \text{where}$$

$z_1 = 1+i = \sqrt{2}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right] = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right] \Rightarrow z_1^8 = 2^2\left[\cos(2\pi)+i\sin(2\pi)\right]$

$z_2 = -1+\sqrt{3}i = 2\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right] = 2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right] \Rightarrow z_2^3 = 2^3\left[\cos(2\pi)+i\sin(2\pi)\right]$

$z_3 = -1-\sqrt{3}i = 2\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right] = 2\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right] \Rightarrow z_3^6 = 2^6\left[\cos(8\pi)+i\sin(8\pi)\right]$

$z_4 = -1+i = \sqrt{2}\left[-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right] = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right] \Rightarrow z_4^4 = 2^2\left[\cos(3\pi)+i\sin(3\pi)\right]$

$\Rightarrow \sqrt{3}\,z_1^8 z_2^3 z_3^6 z_4^4 = 3^{1/2}\cdot 2^4\cdot 2^3\cdot 2^6\cdot 2^2\left[\cancelto{-1}{\cos(15\pi)}+i\cancelto{0}{\sin(15\pi)}\right] = \boxed{-2^{15}3^{1/2}}$

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17) $\dfrac{(-1+\sqrt{3}i)^3}{(-1-\sqrt{3}i)^6}\cdot\dfrac{(1-\sqrt{3}i)^6}{(1+\sqrt{3}i)^3}=$

A) $2^{17}3^{1/2}$ B) -2^{11} C) $2^9(1+\sqrt{3}i)$ $\boxed{\text{D) } -1}$ E) None of the above

$$\frac{(-1+\sqrt{3}i)^3}{(-1-\sqrt{3}i)^6}\cdot\frac{(1-\sqrt{3}i)^6}{(1+\sqrt{3}i)^3}=\frac{z_1^3}{z_2^6}\cdot\frac{z_3^6}{z_4^3}\quad \text{where}$$

$$z_1=-1+\sqrt{3}i=2\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]=2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]\Rightarrow z_1^3=2^3\left[\cos(2\pi)+i\sin(2\pi)\right]$$

$$z_2=-1-\sqrt{3}i=2\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right]=2\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right]\Rightarrow z_2^6=2^6\left[\cos(8\pi)+i\sin(8\pi)\right]$$

$$z_3=1-\sqrt{3}i=2\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right]=2\left[\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right]\Rightarrow z_3^6=2^6\left[\cos(10\pi)+i\sin(10\pi)\right]$$

$$z_4=1+\sqrt{3}i=2\left[\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]=2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]\Rightarrow z_3^3=2^3\left[\cos(\pi)+i\sin(\pi)\right]$$

$$\Rightarrow\frac{(-1+\sqrt{3}i)^3}{(-1-\sqrt{3}i)^6}\cdot\frac{(1-\sqrt{3}i)^6}{(1+\sqrt{3}i)^3}=\frac{\cancelto{1}{2^3\cdot 2^6}}{2^6\cdot 2^3}\left[\cancelto{-1}{\cos(2\pi-8\pi+10\pi-\pi)}+i\cancelto{0}{\sin(2\pi-8\pi+10\pi-\pi)}\right]=\boxed{-1}$$

18) Solve for z: $\quad z^4 + 8 - 8\sqrt{3}\,i = 0$

A) $\pm 1 \pm 3i,\ \pm\sqrt{3} \pm i$ B) $\pm 1 \mp 3i,\ \pm\sqrt{3} \mp i$

C) $\pm 1 \pm 3i,\ \pm\sqrt{3} \mp i$ $\boxed{\text{D) } \pm 1 \mp 3i,\ \pm\sqrt{3} \pm i}$

E) None of the above

$$z^4 + 8 - 8\sqrt{3}i = 0 \Rightarrow z^4 = -8 + 8\sqrt{3}i = 16\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] = 16\left[\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(\frac{2\pi}{3} + 2k\pi\right)\right]$$

$$\Rightarrow z = \left(16\left[\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(\frac{2\pi}{3} + 2k\pi\right)\right]\right)^{1/4} = 16^{1/4}\left[\cos\left(\frac{2\pi}{12} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{2\pi}{12} + \frac{2k\pi}{4}\right)\right]$$

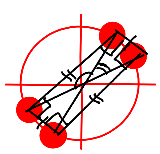
$$= 2\left[\cos\left(\frac{\pi}{6} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + \frac{k\pi}{2}\right)\right]$$

$$\Rightarrow \begin{cases} z_0 = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = 2\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = \sqrt{3} + i \\ z_1 = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = 2\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] = -1 + \sqrt{3}i \\ z_2 = 2\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right] = 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = -\sqrt{3} - i \\ z_3 = 2\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 1 - \sqrt{3}i \end{cases}$$

$$\boxed{\begin{array}{c} \pm 1 \mp \sqrt{3}i \\ \pm\sqrt{3} \pm i \end{array}}$$

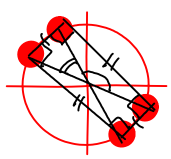
Also, you can tell which of the choices given are not a possibility due to the fact that the solutions will be separated by 90° since we are taking a 4-th root.

A) $\pm 1 \pm 3i,\ \pm\sqrt{3} \pm i$



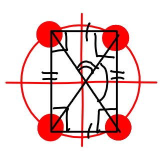
can't be the answer.

B) $\pm 1 \mp 3i,\ \pm\sqrt{3} \mp i$



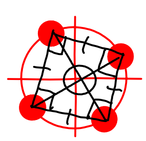
can't be the answer.

C) $\pm 1 \pm 3i,\ \pm\sqrt{3} \mp i$



can't be the answer.

$\boxed{\text{D) } \pm 1 \mp 3i,\ \pm\sqrt{3} \pm i}$



could be the answer because the solutions are equally spaced around a circle.

We verified above that these are our solutions

19) Which of the following is completely true?

A) The equation $a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$ where $a_0, a_1, \ldots, a_4 \in \mathbb{R}$ has 4 solutions that will be real numbers.

B) The equation $a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$ where $a_0, a_4 \in \mathbb{R}$ and $a_3 = a_2 = a_1 = 0$ has 4 solutions that will be real or will come in complex conjugate pairs, they will all have the same radius, and they will be separated by an angle of 90°.

C) The equation $a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$ where $a_0, a_4 \in \mathbb{C}$ and $a_3 = a_2 = a_1 = 0$ has 4 solutions that may not come in complex conjugate pairs and they won't all have the same radius either.

D) All of the above

E) None of the above

A) False. A polynomial equation of degree n with real coefficients will have n solutions that will come in complex conjugate pairs. Any real number is it's own complex conjugate.

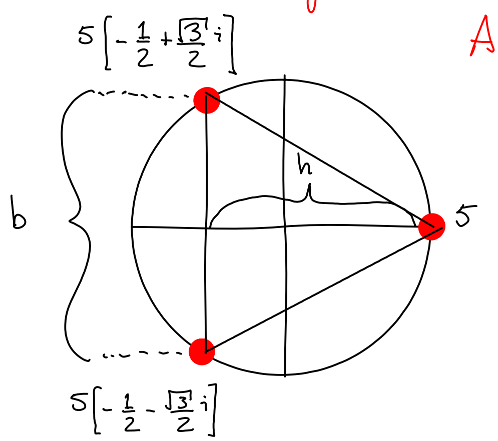
B) True. This equation has the form $a_4 z^4 + a_0 = 0$ or $z^4 = -a_0/a_4$. This is a 4-th degree polynomial so there will be 4 solutions. The coefficients are real so solutions will come in complex conjugate pairs or be real. Since this equation reduced to simply taking a 4-th root, the solutions will be equally spaced around a circle with an angle of $2\pi/4 = \pi/2$ between solutions.

C) False. This equation reduces to $z^4 = -a_0/a_4$ and since we are taking a root, the solutions will fall on a circle, thus have the same radius. Because the coefficients may not be real, the solutions may not come in complex conjugate pairs

20) The solutions to the equation $z^3 + 3 - 4i = 0$ are the vertices (corners) of an equilateral triangle in the complex plane. Find the area of this triangle.

A) 5 B) $\frac{5\sqrt{3}}{2}$ C) $\frac{5}{2}$ D) $\frac{5\pi}{2}$ E) $\boxed{\text{None of the above}}$

$z^3 + 3 - 2i = 0 \Rightarrow z^3 = -3 + 4i$ which has a radius $r = 5$. All three solutions lie on a circle of radius 5 centered at the origin. It doesn't matter where they are because we know there is an angle of $2\pi/3$ between them and they form an equilateral triangle. Pick any three numbers that are equally spread around this circle to form your triangle, the area won't be any different. I'll choose 5, $5\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$, and $5\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$ to make the triangle.



$A = \frac{1}{2}bh$ $b = 5\sqrt{3}$ $h = 5 + \frac{5}{2} = \frac{15}{2}$

$$\Rightarrow A = \frac{1}{2}bh = \frac{1}{2}(5\sqrt{3})\left(\frac{15}{2}\right) = \boxed{\frac{5^2 \cdot 3\sqrt{3}}{4}}$$

21) The graph of $f(x) = 2\sin(3x - \pi) - 4$ looks like the graph of $\sin(x)$ except that the graph of $f(x)$

A) has a period 3 times larger than that of $\sin(x)$

B) has a phase shift of 3π

C) is shifted to the right by 4

D) All of the above

E) None of the above

$f(x) = 2\sin(3x - \pi) - 4 \Rightarrow A = 2,\ \omega = 3,\ \phi = \pi,\ B = -4$

A) False. $f(x)$ has a period $T = 2\pi/\omega = 2\pi/3$ which is 3 times <u>smaller</u> than that of $\sin(x)$

B) False. $f(x)$ has a phase shift of $\phi/\omega = \pi/3 \neq 3\pi$

C) False. $f(x)$ has a shift to the right by $\pi/3$ and down by 4 relative to $\sin x$

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22) The graph of $f(x) = 2\sin(\pi x - 3) - 4$ looks like the graph of $\sin(x)$ except that the graph of $f(x)$

A) has π times more periods than $\sin(x)$

B) has half the amplitude of $\sin(x)$

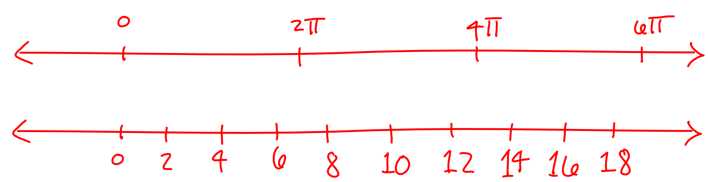
C) has a phase shift of $-\pi/3$

D) All of the above

E) None of the above

$f(x) = 2\sin(\pi x - 3) - 4 \Rightarrow A = 2, \ \omega = \pi, \ \phi = 3, \ B = -4$

A) True. $f(x)$ has a period $T = 2\pi/\omega = 2\pi/\pi = 2$ and $\sin(x)$ has period 2π which is π times that of $f(x)$, therefore there must be π times fewer periods of $\sin(x)$ than $f(x)$ for them to all fit on the real line.



versus

B) False. $f(x)$ has an amplitude of 2 which is twice that of $\sin x$

C) False. $f(x)$ has a phase shift $\phi/\omega = 3/\pi \neq -\pi/3$

23) The graph of $f(x) = 2\sin(\pi x - 3\pi) - 4$ looks like the graph of $\underline{\sin(\pi x)}$ except that the graph of $f(x)$

A) has π times more periods than $\sin(\pi x)$

B) has twice the amplitude of $\sin(x)$

C) has a phase shift of 3

D) All of the above

E) None of the above

$f(x) = 2\sin(\pi x - 3\pi) - 4 \Rightarrow A = 2,\ \omega = \pi,\ \phi = 3\pi,\ B = -4$

A) False. $f(x)$ has a period $T = 2\pi/\omega = 2\pi/\pi = 2$

$\sin(\pi x)$ has a period $T = 2\pi/\omega = 2\pi/\pi = 2$, the same as that of $f(x)$

B) True. $f(x)$ has an amplitude $A = 2$ which is twice that of $\sin x$

C) True. $f(x)$ has a phase shift of $\phi/\omega = 3\pi/\pi = 3$

I'll accept either B or C as correct. Pointing out this error (having 2 correct solutions) was worth extra credit.

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24) Which of the following is completely true?

A) The graph of $\sec x$ can be made by flipping the graph of $\csc x$ about the vertical line $x = \pi/4$

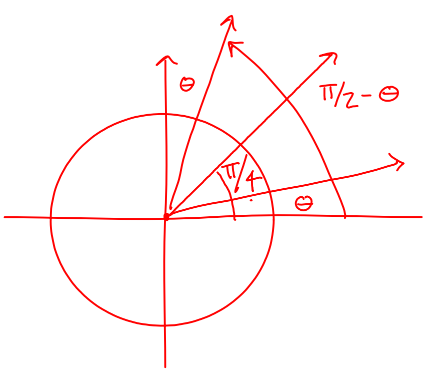
B) The graph of $\cot x$ can be made by flipping the graph of $\tan x$ about the vertical line $x = \pi/4$

C) The graph of $\cos x$ can be made by flipping the graph of $\sin x$ about the vertical line $x = \pi/4$

D) All of the above

E) None of the above

These are all true. You can get the graph of one trig function by flipping the graph of its cofunction over the vertical line $x = \pi/4$. This has to do with the fact that $\text{trig}(\theta) = \text{cotrig}(\pi/2 - \theta)$ and $\pi/2 - \theta$ can be found on the unit circle by flipping the angle θ over the angle $\pi/4$.



25) Which of the following is completely true?

A) There is only 1 triangle that fits the following data: $\alpha = \tan^{-1}(3/4)$, $a=3$, $b=4$

B) There is only 1 triangle that fits the following data: $\alpha = \sin^{-1}(3/5)$, $a=3$, $b=4$

C) $\boxed{\text{There is only 1 triangle that fits the following data: } \beta = \sin^{-1}(4/5),\ a=3,\ b=4}$

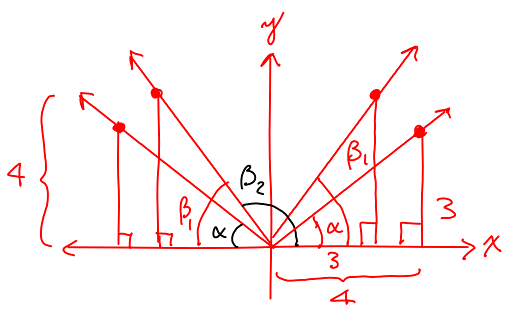
D) All of the above

E) None of the above

A) $\sin\beta = \frac{b}{a}\sin\alpha = \frac{4}{3}\sin(\tan^{-1}(3/4)) = \frac{4}{3}\sin\alpha$ where $\alpha = \tan^{-1}(3/4)$

$\Rightarrow \tan\alpha = 3/4$ and $\cot^2\alpha + 1 = \csc^2\alpha = \frac{1}{\sin^2\alpha} \Rightarrow \sin\alpha = \frac{1}{\sqrt{1+\cot^2\alpha}} = \frac{1}{\sqrt{1+16/9}}$

$= \frac{1}{\sqrt{25/9}} = \frac{3}{5} \Rightarrow \sin\beta = \frac{4}{3}\sin\alpha = \frac{4}{3}\cdot\frac{3}{5} = \frac{4}{5} \Rightarrow \beta_1 = \sin^{-1}\left(\frac{4}{5}\right)$



$\beta_2 = \pi - \beta_1 = \pi - \sin^{-1}\left(\frac{4}{5}\right)$

$\tan\alpha = 3/4 = y_1/x_1$

$\sin\beta_1 = 4/5 = y_2/r_2 \qquad x_2 = \sqrt{r_2^2 - y_2^2} = \sqrt{25-16} = 3$

You can see from the picture that $\alpha + \beta_2 < 180°$ $\Rightarrow$ There is a 2nd solution.

B) $\sin\beta = \frac{b}{a}\sin\alpha = \frac{4}{3}\sin(\sin^{-1}(3/5)) = \frac{4}{3}\sin\alpha$ where $\alpha = \sin^{-1}(3/5)$

$\Rightarrow \sin\alpha = 3/5 \Rightarrow \sin\beta = \frac{b}{a}\sin\alpha = \frac{4}{3}\cdot\frac{3}{5} = \frac{4}{5} \Rightarrow \beta_1 = \sin^{-1}(4/5)$

and $\beta_2 = \pi - \sin^{-1}(4/5)$. We'll draw another picture to see if β_2 provides an actual solution.

$\sin\beta_1 = 4/5 = y_1/r_1 \Rightarrow x_1 = \sqrt{r_1^2 - y_1^2} = \sqrt{25-16} = 3 \Rightarrow \beta_1$ corresponds to $(3,4)$ and

$\beta_2 = \pi - \beta_1 \Rightarrow \beta_2$ corresponds to $(-3,4)$

$\sin\alpha = 3/5 = y_2/r_2 \Rightarrow x_2 = \sqrt{r_2^2 - y_2^2} = \sqrt{25-9} = 4 \Rightarrow \alpha$ corresponds to $(4,3)$

This gives the same diagram as in part A. $\Rightarrow$ There is a 2nd solution.

This shouldn't be a surprise. The data given here is equivalent to that given in part A,

$a=3$, $b=4$ for both A and B and $\alpha = \tan^{-1}(3/4) \Rightarrow \sin\alpha = 3/5$ or $\alpha = \sin^{-1}(3/5)$

c) $\beta = \sin^{-1}(4/5) \Rightarrow \sin\beta = 4/5$

$\Rightarrow \sin\alpha = \frac{a}{b}\sin\beta = \frac{3}{4}\left(\frac{4}{5}\right) = \frac{3}{5} \Rightarrow \alpha_1 = \sin^{-1}(3/5)$

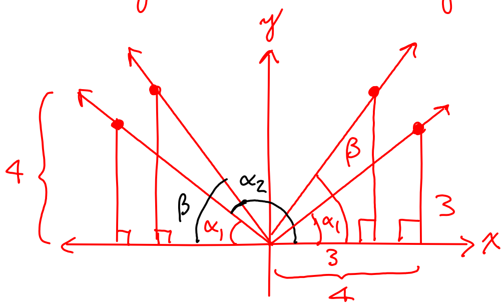
We'll draw another diagram if necessary.

$\sin\alpha_1 = 3/5 = y_1/r_1 \Rightarrow x_1 = \sqrt{r_1^2 - y_1^2} = \sqrt{25-9} = 4 \Rightarrow \alpha_1$ corresponds to $(4,3)$

$\alpha_2 = \pi - \alpha_1 \Rightarrow \alpha_2$ corresponds to $(-4,3)$

$\sin\beta = 4/5 = y_2/r_2 \Rightarrow x_2 = \sqrt{r_2^2 - y_2^2} = \sqrt{25-16} = 3 \Rightarrow \beta$ corresponds to $(3,4)$

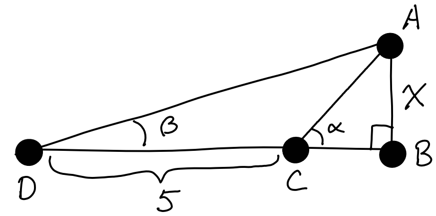
This gives the same diagram with the labels changed in an important way



You can see from the picture that

$\alpha_2 + \beta > 180° \Rightarrow$ There is only one solution

26) If $\alpha = 50°$ and $\beta = 20°$, find x



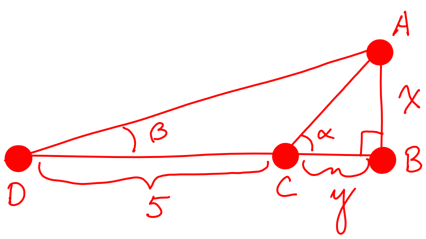
A) $\dfrac{5\tan(50°)\tan(20°)}{\tan(50°)+\tan(20°)}$

B) $\dfrac{5\tan(50°)\tan(20°)}{\tan(20°)-\tan(50°)}$

C) $\boxed{\dfrac{5\tan(50°)\tan(20°)}{\tan(50°)-\tan(20°)}}$

D) $5[\tan(70°)-\tan(50°)+\tan(20°)]$

E) None of the above



$\tan(50°) = \dfrac{x}{y} \Rightarrow y = \dfrac{x}{\tan(50°)}$

$\tan(20°) = \dfrac{x}{5+y} = \dfrac{x}{5+\dfrac{x}{\tan(50°)}} = \dfrac{x\tan(50°)}{x+5\tan(50°)}$

$\Rightarrow \tan(20°)[x+5\tan(50°)] = x\tan(50°)$

$\Rightarrow x[\tan(20°)-\tan(50°)] = -5\tan(50°)\tan(20°)$

$\Rightarrow \boxed{x = \dfrac{5\tan(50°)\tan(20°)}{\tan(50°)-\tan(20°)}}$

27) Which of the following is completely true?

A) If $a=1$, $\beta=30°$, and $\gamma=50°$ are given then $b=\dfrac{\sin(30°)}{\sin(100°)}$

B) If $a=1$, $b=2$, and $\gamma=50°$ are given then $c=\sqrt{5-4\cos(50°)}$

C) If $a=1$, $b=2$, and $\gamma=50°$ are given then $\beta=\sin^{-1}\left[\dfrac{2\sin(50°)}{\sqrt{5-4\cos(50°)}}\right]$

$\boxed{\text{D) All of the above}}$

E) None of the above

A) True. $\alpha+\beta+\gamma=180° \Rightarrow \alpha=100° \Rightarrow b=\dfrac{a\sin\beta}{\sin\alpha}=\dfrac{\sin(30°)}{\sin(100°)}$

B) True. $c^2=a^2+b^2-2ab\cos\gamma \Rightarrow c=\sqrt{1+4-4\cos(50°)}=\sqrt{5-4\cos(50°)}$

C) True. $\sin\beta=\dfrac{b\sin\gamma}{c}=\dfrac{2\sin(50°)}{\sqrt{5-4\cos(50°)}} \Rightarrow \beta=\sin^{-1}\left[\dfrac{2\sin(50°)}{\sqrt{5-4\cos(50°)}}\right]$

EXTRA CREDIT PROBLEMS 28-30

28) $r[\cos\theta \pm i\sin\theta] = re^{\pm i\theta}$ in complex exponential notation. What is $\tan\theta$?

A) $\dfrac{e^{i\theta}+e^{-i\theta}}{2}$ B) $\dfrac{e^{i\theta}+e^{-i\theta}}{2i}$ C) $\dfrac{e^{i\theta}-e^{-i\theta}}{2}$

D) $\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ $\boxed{\text{E) None of the above}}$

$\cos\theta + i\sin\theta + \cos\theta - i\sin\theta = e^{i\theta} + e^{-i\theta}$

$\Leftrightarrow\ 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$\Leftrightarrow\ \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\cos\theta + i\sin\theta - [\cos\theta - i\sin\theta] = e^{i\theta} - e^{-i\theta}$

$\Leftrightarrow\ 2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$\Leftrightarrow\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\Rightarrow\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\dfrac{e^{i\theta}-e^{-i\theta}}{2i}}{\dfrac{e^{i\theta}+e^{-i\theta}}{2}} = \frac{e^{i\theta}-e^{-i\theta}}{i[e^{i\theta}+e^{-i\theta}]}$$

29) $\log_{10}\left(x^2\sqrt{x^3+1}\right) = \log_{10}(x^2) + \log_{10}(\sqrt{x^3+1}) = 2\log_{10}(x) + (1/2)\log_{10}(x^3+1)$

$\boxed{\text{A) } 2\log_{10}(x) + \dfrac{\log_{10}(x^3+1)}{2}}$

B) $2(1/2)(3)\left[\log_{10}(x) + \log_{10}(\sqrt{x^3+1})\right]$

C) $\log_{10}(x^2)\cdot\log_{10}(\sqrt{x^3+1})$

D) All of the above

E) None of the above

30) Solve for x: $2^{x+1}\cdot 16^{-x} = 1/2$

A) $x = 3/2$ $\boxed{\text{B) } x = 2/3}$ C) $x = 0$ D) $x = \log_2$ E) None of the above

$2^{x+1}\cdot 16^{-x} = 1/2 \Leftrightarrow 2^{x+1}\cdot 2^{-4x} = 2^{-3x+1} = 2^{-1} \Leftrightarrow -3x+1 = -1$

$\Leftrightarrow\ x = 2/3$