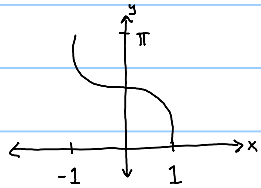
EXAM #3 Solutions

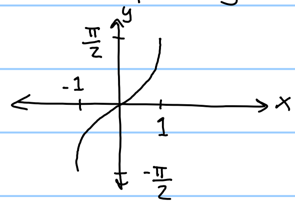
Note Title 6/27/2006

#1 Which of the following are completely true?

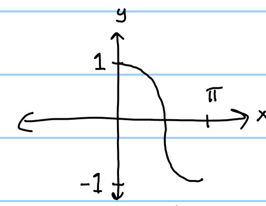
E

a) 

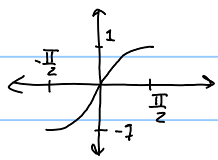
is a graph of $\sin^{-1}(x)$

b) 

is a graph of $\cos^{-1}(x)$

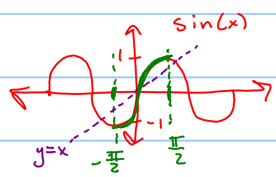
c) 

is a graph of $\cos^{-1}(x)$

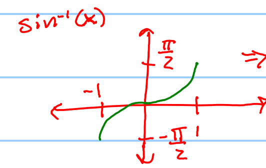
d) 

is a graph of $\sin^{-1}(x)$

e) None of the above

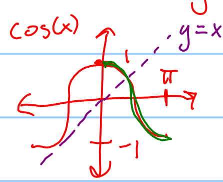
a) 

switch domain and range or fold over line $y=x$

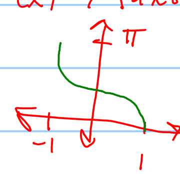


$\Rightarrow$ false

b) This is the graph of $\sin^{-1}(x)$, not $\cos^{-1}(x)$ $\Rightarrow$ false

c) 

switch domain and range or fold over line $y=x$



$\Rightarrow$ false

just look at range of given graph and you should know this is not $\cos^{-1}(x)$

d) false. see (a)

$\Rightarrow$ e) is true

#2 Which of the following are completely true?

C

a) $\mathbb{R} \to \sin(x) \to [-1,1] \to \arcsin(x) \to [0,\pi]$

b) $\mathbb{R} \to \sin(x) \to (-1,1) \to \arcsin(x) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

c) $\mathbb{R} \to \cos(x) \to [-1,1] \to \arccos(x) \to [0,\pi]$

d) $[-1,1] \to \sin(x) \to [0,\pi] \to \arcsin(x) \to \mathbb{R}$

e) None of the above

a) false $\arcsin(x) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

b) false $\sin(x) \to [-1,1] \to \arcsin(x)$

c) true

d) false $\mathbb{R} \to \sin(x) \to [-1,1] \to \sin^{-1}(x) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

#3 Which of the following are completely true?

B

a) $\sin^{-1}(\sin(\frac{7\pi}{4})) = \frac{7\pi}{4}$ because $\frac{7\pi}{4}$ is on the right half of the unit circle

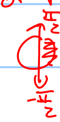
b) $\sin^{-1}(\sin(\frac{7\pi}{4})) = -\frac{\pi}{4}$ because $\sin(\frac{7\pi}{4}) = \sin(-\frac{\pi}{4})$ and the range of $\sin^{-1}(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

c) $\sin(\sin^{-1}(\frac{\pi}{2})) = \frac{\pi}{2}$

d) $\sin(\sin^{-1}(\frac{\sqrt{3}}{2})) = 60°$

e) None of the above

a) false. When I refer to the "right half of the unit circle", I mean the interval from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$



In fact,

$\sin^{-1}(\sin(\frac{7\pi}{4})) = \sin^{-1}(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}$

b) true

c) false. $\sin^{-1}(\frac{\pi}{2}) = \sin^{-1}(\frac{3.14..}{2})$ is undefined since $\frac{\pi}{2} > 1$

$\sin^{-1}(\frac{\pi}{2})$ means "sin of what angle gives $\frac{\pi}{2}$?". $\frac{\pi}{2} > 1$ and $-1 < \sin\theta < 1$ so $\sin^{-1}(\frac{\pi}{2})$ is undefined.

d) false. $\sin(\sin^{-1}(\frac{\sqrt{3}}{2}))$ = a number, not an angle. Although,

$\sin^{-1}(\sin(60°)) = \sin^{-1}(\frac{\sqrt{3}}{2}) = 60°$

#4 Which of the following are completely true?

BC

a) $\cos(\cos^{-1}(\frac{\pi}{2})) = \frac{\pi}{2}$

b) $\cos(\cos^{-1}(\frac{\sqrt{3}}{2})) = \frac{\sqrt{3}}{2}$

c) $\cos^{-1}(\cos(\frac{9\pi}{4})) = \frac{\pi}{4}$

d) $\cos^{-1}(\cos(-\frac{\pi}{4})) = -\frac{\pi}{4}$

e) None of the above

a) $\cos(\cos^{-1}(\frac{\pi}{2}))$ = undefined because $\frac{\pi}{2} > 1 \Rightarrow$ false

b) $\cos(\cos^{-1}(\frac{\sqrt{3}}{2})) = \frac{\sqrt{3}}{2}$ because $\frac{\sqrt{3}}{2} \in [-1, 1] \Rightarrow$ true

c) $\cos^{-1}(\cos(\frac{9\pi}{4})) = \cos^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4} \Rightarrow$ true

d) $\cos^{-1}(\cos(-\frac{\pi}{4})) = \cos^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4} \neq -\frac{\pi}{4} \Rightarrow$ false

#5 Which of the following are completely true?

a) If $f(x) = y$ has one output per one input then it passes the vertical line test and is a function.

AC

b) If a function has multiple inputs that give the same output then it passes the horizontal line test and has an inverse

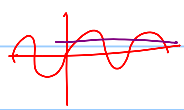
c) If a function passes the horizontal line test, then it has an inverse that can be seen by folding the graph of $f(x)$ over the line $y = x$.

d) Only $\tan(x)$ has a perfect unrestricted inverse

e) None of the above

a) true

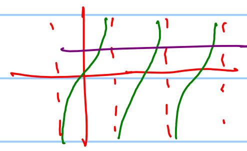
b) false. Consider $\sin(x)$: $\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \sin\left(\frac{9\pi}{4}\right) = \cdots$



doesn't pass horizontal line test + doesn't have a real inverse.

c) true

d) false



doesn't pass horizontal line test

#6 Which of the following are completely true?

a) $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{\sqrt{2}}{4}$

b) $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{1}{2\sqrt{2}}$

ABD

c) $\sin^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right) = \cos^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$

d) $\cot^{-1}(1) = \tan^{-1}(1)$

e) None of the above

a) $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \tan(\alpha)$ where $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$

$\Rightarrow \sin\alpha = \frac{1}{3}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

let $y = 1$ and $r = 3$ then $x = \pm\sqrt{9-1} = \pm\sqrt{8} = \pm 2\sqrt{2}$

$\Rightarrow \tan\alpha = y/x = \frac{1}{\pm 2\sqrt{2}}$ but $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\alpha = \frac{1}{3} > 0$

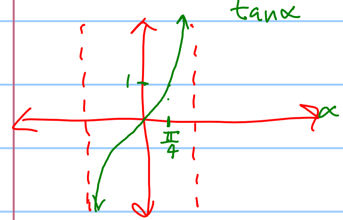
$\Rightarrow \tan\alpha > 0 \Rightarrow \tan\alpha = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4} \Rightarrow$ true

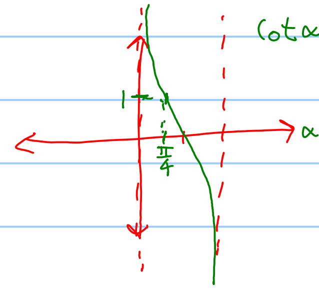
b) true. see (a)

c) $\sin^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ but

$\cos^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \Rightarrow$ false

d) $\cot^{-1}(1) = \alpha \Rightarrow \cot\alpha = 1 \Rightarrow \tan\alpha = \frac{1}{1} = 1 \Rightarrow \tan^{-1}(1) = \alpha$





$\tan\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1 \Rightarrow \cot^{-1}(1) = \tan^{-1}(1) = \frac{\pi}{4} \Rightarrow$ true

#7 Which of the following are completely true?

A

a) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{\sqrt{5}}$

b) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{-\sqrt{5}}{5}$

c) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = -\frac{1}{\sqrt{5}}$

d) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{5}$

e) None of the above

a) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \sin\alpha$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \tan\alpha = \frac{1}{2} = \frac{y}{x}$ let $y=1$, $x=2$ then $r = \sqrt{4+1} = \sqrt{5}$

$\Rightarrow \sin\alpha = y/r = \pm 1/\sqrt{5}$ but $\tan\alpha = \frac{1}{2} > 0$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin\alpha = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{\sqrt{5}}{5} \Rightarrow$ true

b) false. see (a)

c) false. see (a)

d) false. see (a)

#8 $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} =$

ABC

a) $2\csc\theta$ b) $\frac{2}{\sin\theta}$ c) $\frac{2\cot\theta}{\cos\theta}$ d) $\frac{1}{2}\csc\theta$ e) None of the above

$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{(1+\cos\theta)} \cdot \frac{\sin\theta}{\sin\theta} + \frac{(1+\cos\theta)}{\sin\theta} \cdot \frac{(1+\cos\theta)}{(1+\cos\theta)}$

$= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} + \frac{(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$

$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} = \frac{\overbrace{\sin^2\theta + \cos^2\theta}^{1} + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1 + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}} = \frac{2}{\sin\theta} = 2\csc\theta$$

$\Downarrow$ (b) true $\quad \Downarrow$ (a) true

c) $2\frac{\cot\theta}{\cos\theta} = 2\frac{\cancel{\cos\theta}}{\sin\theta}\cdot\frac{1}{\cancel{\cos\theta}} = \frac{2}{\sin\theta} \Rightarrow$ true

d) $\frac{1}{2}\csc\theta \neq 2\csc\theta \Rightarrow$ false

#9 $(1-\cos^2\theta)(1+\cot^2\theta) =$

AC

a) $\sin\theta\csc\theta$ b) $\sin^2\theta$ c) 1 d) $\cos^2\theta$ e) None of the above

$$\overbrace{(1-\cos^2\theta)}^{\sin^2\theta}(1+\cot^2\theta) = \sin^2\theta(1+\cot^2\theta) = \sin^2\theta + \sin^2\theta\cot^2\theta$$

$$= \sin^2\theta + \sin^2\frac{\cos^2\theta}{\sin^2\theta} = \sin^2\theta + \cos^2\theta = 1 \Rightarrow \text{(c) is true}$$

a) $\sin\theta\csc\theta = \sin\theta\cdot\frac{1}{\sin\theta} = 1 \Rightarrow$ true

b) $\sin^2\theta \neq 1 \Rightarrow$ false $\quad$ d) $\cos^2\theta \neq 1 \Rightarrow$ false

#10 $1 - \frac{\sin^2\theta}{1+\cos\theta} =$

BD

a) $\sin\theta$ b) $\cos\theta$ c) $\frac{1}{\csc\theta}$ d) $\frac{\tan\theta\cot\theta}{\sec\theta}$ e) None of the above

$$1 - \frac{\sin^2\theta}{1+\cos\theta} = \frac{\overbrace{1}^{1}+\cos\theta}{1+\cos\theta} - \frac{\sin^2\theta}{1+\cos\theta} = \frac{1+\cos\theta-\sin^2\theta}{1+\cos\theta}$$

$$= \frac{\overbrace{1-\sin^2\theta}^{\cos^2\theta}+\cos\theta}{1+\cos\theta} = \frac{\cos^2\theta+\cos\theta}{1+\cos\theta} = \frac{\cos\theta\cancel{(1+\cos\theta)}}{\cancel{1+\cos\theta}} = \cos\theta$$

$\Rightarrow$ (b) true

a) $\sin\theta \neq \cos\theta \Rightarrow$ false

c) $\frac{1}{\csc\theta} = \frac{1}{\frac{1}{\sin\theta}} = \sin\theta \Rightarrow$ false

d) $\dfrac{\tan\theta\cot\theta}{\sec\theta} = \dfrac{\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}} = \dfrac{1}{\frac{1}{\cos\theta}} = \cos\theta \Rightarrow$ true

#11 $\dfrac{\sec\theta}{\csc\theta} + \dfrac{\sin\theta}{\cos\theta} =$

BC

a) $2\cot\theta$ b) $2\tan\theta$ c) $(2\sin^2\theta + 2\cos^2\theta)\left(\frac{\sin\theta}{\cos\theta}\right)$ d) 1

e) None of the above

$$\frac{\sec\theta}{\csc\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}} + \frac{\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{1} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 2\frac{\sin\theta}{\cos\theta} = 2\tan\theta \Rightarrow \text{(b) true}$$

a) $2\cot\theta \neq 2\tan\theta \Rightarrow$ false

c) $(2\sin^2\theta + 2\cos^2\theta)\left(\frac{\sin\theta}{\cos\theta}\right) = 2\underbrace{(\sin^2\theta + \cos^2\theta)}^{1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$= 2\frac{\sin\theta}{\cos\theta} = 2\tan\theta \Rightarrow$ true

d) $1 \neq 2\tan\theta \Rightarrow$ false

#12 $\dfrac{1+\tan\theta}{1+\cot\theta} =$

BC

a) $\cot\theta$ b) $\dfrac{1}{\left(\frac{1}{\tan\theta}\right)}$ c) $\tan\theta$ d) $\dfrac{1}{\left(\frac{1}{\cot\theta}\right)}$ e) None of the above

$$\frac{1+\tan\theta}{1+\cot\theta} = \frac{1+\frac{\sin\theta}{\cos\theta}}{1+\frac{\cos\theta}{\sin\theta}} = \frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}} = \frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\cos\theta+\sin\theta}{\sin\theta}}$$

$$= \left(\frac{\cancel{\cos\theta+\sin\theta}}{\cos\theta}\right)\left(\frac{\sin\theta}{\cancel{\cos\theta+\sin\theta}}\right) = \frac{\sin\theta}{\cos\theta} = \tan\theta \Rightarrow \text{(c) true}$$

a) $\cot\theta \neq \tan\theta \Rightarrow$ false

b) $\dfrac{1}{\left(\frac{1}{\tan\theta}\right)} = \dfrac{1}{\cot\theta} = \tan\theta \Rightarrow$ true

d) $\dfrac{1}{\left(\frac{1}{\cot\theta}\right)} = \dfrac{1}{\tan\theta} = \cot\theta \neq \tan\theta \Rightarrow$ false

#13 Extra Credit

The set of even numbers has fundamentally different properties then the odd numbers because

a) The odd numbers are closed under addition

b) The even numbers have an additive identity but the odd numbers don't

c) There are twice as many even numbers as odd numbers because all even numbers are divisible by 2 but all odd numbers are only divisible by 1 and 2 is twice as much as 1.

d) Only the evens have an additive inverse.

e) None of the above

odd odd even

a) false. Consider 3+5=8

To be "closed" under addition means adding any two of a given set will give you something within that same set.

b) True: The additive identity is zero. The identity is the element that doesn't change any other element in the set under the given operation. For example:

$1+0=1, \quad 2+0=2, \quad 8+0=8 \ldots$

c) False. There are countably infinite evens and odds. All evens are divided by 2 though and the only common divisor of all odds is 1. This is why 2 is the only even prime number.

d) To have an inverse means there is an element in the set so that if you put the two elements together under the given operation, you get the identity. For example

$2+(-2)=0, \quad 1+(-1)=0, \quad 8+(-8)=0, \ldots$

Thus the evens have an additive inverse.

Although $3+(-3)=0$, $5+(-5)=0, \ldots$ would show the inverse property for odds except zero isn't an odd number. $\Rightarrow$ true

KEY

#1. E

2. C

3. B

4. B, C

5. A, C

6. A, B, D

7. A

8. A, B, C

9. A, C

10. B, D

11. B, C

12. B, C

13. B, D