Introduction To Complex Numbers

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$$i = \sqrt{-1}$$

Summary

Complex number standard or rectangular form: $z = x + yi$

polar form: $z = r[\cos\theta + i\sin\theta]$

exponential form: $z = re^{i\theta}$

$x = r\cos\theta$

$y = r\sin\theta$

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ then

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

$$z_1 \cdot z_2 = x_1 x_2 + y_1 y_2 i^2 + x_1 y_2 i + x_2 y_1 i = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \left(\frac{x_1 + y_1 i}{x_2 + y_2 i}\right)\left(\frac{x_2 - y_2 i}{x_2 - y_2 i}\right) = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{(x_2 y_1 - x_1 y_2)i}{x_2^2 + y_2^2}$$

$z_1 = r_1[\cos\theta_1 + i\sin\theta_1]$ $\quad z_2 = r_2[\cos\theta_2 + i\sin\theta_2]$

$$z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$z_1 = r_1 e^{i\theta_1}$ $\quad z_2 = r_2 e^{i\theta_2}$ $\Rightarrow$ $z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

If $z = x + iy = r[\cos\theta + i\sin\theta] = re^{i\theta}$ then its

Complex conjugate is $\bar{z} = x - iy = r[\cos\theta - i\sin\theta] = re^{-i\theta}$

[replace i with $-i$]
or
[reflect z over Real axis]

De Moivre's Theorem: $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right)\right], \quad K = 0, 1, \ldots, n-1$$

Part One: Introduction

If we want to solve for x in the equation $x^2+1=0$ we get:

$$x^2+1=0 \Leftrightarrow x^2=-1 \Leftrightarrow x=\pm\sqrt{-1}$$

This might not make sense at first because any real number squared is positive.

$1^2=1 \quad (-1)^2=(-1)(-1)=1 \quad (-2)^2=(-2)(-2)=4 \quad$ etc.

So what number times itself is -1? Well, it is $\sqrt{-1}$ since $(\sqrt{-1})^2=-1$.

Since $\sqrt{-1}$ can't be a real number we call it an **imaginary number** and denote it with the letter i:

$$\boxed{\sqrt{-1}=i}$$

The term "imaginary" is unfortunate because $\sqrt{-1}$ is just as much a number as any other, it is just that the name "real numbers" was already taken.

The square root of any number can be written with the number i:

$\sqrt{-4}=\sqrt{4}\sqrt{-1}=2i \qquad \sqrt{-8}=\sqrt{8}\sqrt{-1}=\sqrt{8}\,i=\sqrt{2\cdot2\cdot2}\,i=2\sqrt{2}\,i \quad$ etc.

Remember Your Algebra:

$\sqrt{x}=x^{1/2}$ and $(a\cdot b)^n=a^n\cdot b^n$ so $\sqrt{-4}=(-4)^{1/2}=[(4)(-1)]^{1/2}=(4)^{1/2}(-1)^{1/2}=\sqrt{-1}\sqrt{4}=2i$

And your roots: $\sqrt[3]{x}=x^{1/3}$ so $\sqrt[3]{8}=8^{1/3}=2$ since $2^3=8$:

$\sqrt[3]{8}=\sqrt[3]{2\cdot2\cdot2}=2$, it takes 3 two's inside a cube root to be able to pull one 2 out.

Similarly, $\sqrt[4]{48}=48^{1/4}=\sqrt[4]{2\cdot2\cdot2\cdot2\cdot3}=2\sqrt[4]{3}$

Also note that $\sqrt{2}$ is a positive number. If you are solving the equation $x^2=2$ then the solutions are $\sqrt{2}$ and $-\sqrt{2}$. Sometimes beginners get confused because they are used to seeing $\pm\sqrt{2}$ and think that $\sqrt{2}$ is both positive and negative. This is not the case. $\sqrt{2}$ is positive. $-\sqrt{2}$ is negative.

Any real number times $\sqrt{-1}$ is an imaginary number. Remember the symbol $\mathbb{R}$ denotes all real numbers and the symbol "$\in$" means "**an element of**", so the number

xi where $x\in\mathbb{R}$ is an imaginary number.

We know $(\sqrt{x})^2 = x$ from basic algebra. The same works for imaginary numbers, including i:

$\boxed{i = \sqrt{-1}} \Leftrightarrow i^2 = (\sqrt{-1})^2 = \boxed{-1 = i^2} \Leftrightarrow i^3 = i^2 \cdot i = \boxed{-i = i^3} \Leftrightarrow i^4 = i^2 \cdot i^2 = (-1)(-1) = \boxed{1 = i^4}$

So anytime you see i^2, think -1.

How do we add or subtract a real number to or from an imaginary one?

The same way we do it with square roots without using decimals:

$3+\sqrt{2}$ is left written as $3+\sqrt{2}$

So, $3+2\sqrt{-1} = 3+2i$ is left written as $3+2i$.

Our analogy here falls a bit short because $\sqrt{2}$ can be written as a decimal and then you can combine the 3 and the $\sqrt{2}$ to get a single decimal number. It doesn't work this way when combining real and imaginary numbers, we just have to leave them separate.

Numbers of the form $3+2i$ are called "Complex" numbers because they have a real part and an imaginary part. Together they form a single complex number. In this example, 3 is the real part and the imaginary part is 2.

We often use the letter x and y to represent an unknown real number and the letter z to denote an unknown Complex number. Sometimes we also use the letter z if we need a 3rd variable like when we do math in 3 dimensions.

Any letter or symbol can be used to represent pretty much anything you want. However, there is usually a standard "convention" like using theta, denoted θ, to represent an angle. This is done so your brain will quickly associate a symbol with a certain concept. We have to keep an eye out for exceptions to the convention by being aware of the context that the symbol is being used in.

Complex numbers are often denoted by the letter z and their STANDARD FORM looks like this:

$\boxed{z = x+yi \text{ where } x, y \in \mathbb{R}}$ This form is also called RECTANGULAR FORM.

So, again, a complex number is a real number plus a real number times $i = \sqrt{-1}$.

In fact, all of our numbers are complex numbers. Remember, zero is a real number. So,

$3 = 3 + 0i$ $\qquad 2i = 0 + 2i$ $\qquad \pi = 3.1415.... = \pi + 0i$ $\qquad e = 2.71... = e + 0i$

Numbers that don't have an imaginary part (their imaginary part is zero) are called

PURELY REAL COMPLEX NUMBERS and

Numbers that don't have a real part (their real part is zero) are called

PURELY IMAGINARY COMPLEX NUMBERS.

Just as we use the symbol $\mathbb{R}$ to denote the set of all real numbers, we use the symbol $\mathbb{C}$ to denote the set of all complex numbers.

COMPLEX ARITHEMATIC IN RECTANGULAR / STANDARD FORM

Adding, subtracting, and multiplying complex numbers is fairly straight forward.

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ then

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

$$z_1 \cdot z_2 = x_1 x_2 + y_1 y_2 i^2 + x_1 y_2 i + x_2 y_1 i = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \left(\frac{x_1 + y_1 i}{x_2 + y_2 i}\right)\left(\frac{x_2 - y_2 i}{x_2 - y_2 i}\right) = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{(x_2 y_1 - x_1 y_2)i}{x_2^2 + y_2^2}$$

This looks a lot scarier than it really is. Let's look at an example.

If $z_1 = 3 - 2i$ and $z_2 = 4 + i$ then

$$z_1 + z_2 = (3 - 2i) + (4 + i) = (3 + 4) + (-2i + i) = 7 - i$$

$$z_1 - z_2 = (3 - 2i) - (4 + i) = (3 - 4) + (-2i - i) = -1 - 3i$$

$$z_1 \cdot z_2 = (3 - 2i)(4 + i) = 12 + 3i - 8i - 2i^2 = 12 + 2 - 5i = 14 - 5i$$

Just collect the real and imaginary parts. You can treat i as if it was x except $i^2 = -1$

Just multiply like you normally would.

Division requires a trick. How do we get the answer into rectangular / standard form? Use a "smart one."

$$\frac{z_1}{z_2} = \frac{3 - 2i}{4 + i} = \left(\frac{3 - 2i}{4 + i}\right)\left(\frac{4 - i}{4 - i}\right) = \frac{12 - 3i - 8i + 2i^2}{4^2 - 4i + 4i - i^2} = \frac{10 - 11i}{17} = \frac{10}{17} - \frac{11}{17}i$$

We chose our "smart one" to be $4 - i$ because $(4+i)(4-i) = 4^2 + 1^2 = 17$, a real number.

In general, if $z = x + yi$, then $\bar{z} = x - yi$ is called its complex conjugate.

Sometimes a $*$ symbol is used instead, $\bar{z} = z^*$. We have the nice property,

$$z\bar{z} = (x + yi)(x - yi) = x^2 - xyi + xyi - y^2 i^2 = x^2 + y^2$$ which is a real number.

It was this property that allowed us to put z_1/z_2 into standard form.

We'll see more of the complex conjugate later.