Part Two: Visualizing Number Systems

The Number System

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of all "Natural" numbers, also known as the "Counting" Numbers.

Sometimes zero is left out. It depends on who you ask.

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of all "Integers." Notice the Natural numbers are a subset of the integers. This is stated symbolically by, $\mathbb{N} \subset \mathbb{Z}$. It's like a "less than" sign but rounded and used for sets.

$\mathbb{Q} = \left\{\ldots, \frac{-2}{3}, \frac{-2}{2}, \frac{-2}{1}, \frac{-1}{3}, \frac{-1}{2}, \frac{-1}{1}, 0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{4}{4}, \ldots, \frac{8}{4}, \ldots\right\}$ is the set of all "Rational" numbers,

also known as the "Rationals". These are the numbers that can be represented by an integer over an integer. This is stated symbolically by,

$\mathbb{Q} = \left\{\frac{n}{m} : n, m \in \mathbb{Z}\right\}$ read out loud as, "the set of all n over m, such that n and m are integers."

Notice that the integers are a subset of the rationals. Or, symbolically, $\mathbb{Z} \subset \mathbb{Q}$.

Notice how there is an infinite number of ways to represent any number:

$$1 = \frac{2}{2} = \frac{3}{3} = \cdots, \quad \frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \cdots$$

The rationals also have the special property that when you write one as a decimal, it has a repeating pattern.

$$1 = 1.000\ldots, \quad \frac{2}{3} = 0.6666\ldots, \quad \frac{25}{16} = 1.56250000\ldots, \quad \frac{13}{11} = 1.18181818\ldots$$

A standard notation to denote a repeating decimal is to put a bar over the repeating sequence:

$$\frac{13}{11} = 1.18181818\ldots = 1.\overline{18}$$

The repeating sequence could be a billion numbers in a row that don't exhibit any pattern, and then suddenly, that sequence of those billion numbers will start over.

$\mathbb{R}$ = {all the numbers with zero for their imaginary part} are the "Real" numbers.

These include your rationals and all the numbers that don't have a repeating decimal form. These "nonrational" numbers are called "Irrational".

They cannot be expressed as a fraction of integers. So, the real numbers are the set of numbers you get when you combine the rationals with the irrationals. Thus the rationals are a subset of the real numbers. Stated symbolically, $\mathbb{Q} \subset \mathbb{R}$.

$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ are our "COMPLEX" numbers. We've seen that the real numbers are a subset of the complex numbers, $\mathbb{R} \subset \mathbb{C}$

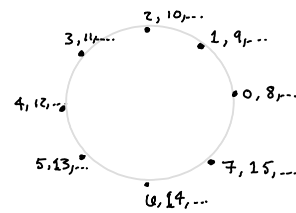
Overall, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

When we have several items, books, baseball cards, etc., we like to put them into some kind of order so we can easily find the one we are looking for, and also to make them look nice.

THE REAL LINE

If we only had the natural numbers, we could put them in order from least to greatest so that we could easily find the one we want, or at least visualize how a particular natural number is related to all the other natural numbers. We could let a dot represent each natural number.

We could put these dots around a circle, but that might hurt our eyes because the dots would start to overlap one another once we'd gone all the way around the circle.



Or we could just put them in order from left to right:

0 1 2 3 4 5 6 7 ..... etc.

Maybe they would look best if we did this along a straight line and equally spaced:

0 1 2 3 4 5 6 7 8 9 10 11 . . . . etc

That looks a lot better.

So, we have a nice dotted line that goes forever to the right and prodives a good visual representation of the natural numbers.

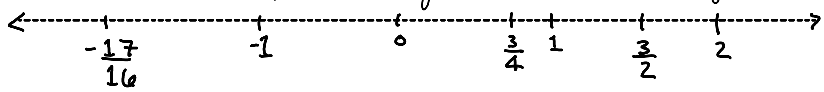
If we extend this idea to the integers we get a dotted line that is twice as long.

If there is an infinite number of natural numbers, and there are twice as many integers as there are natural numbers, then we should have $2 \cdot \infty$ numbers, right?

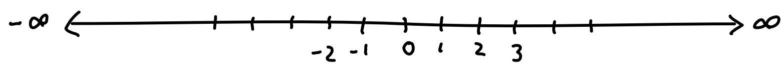
Yes and no because $2 \cdot \infty = \infty$. Remember, infinity is not a number, it is the concept of limits. You'll get more into infinity and its uses in Calculus. Now our dotted line goes forever both ways.

etc . . . . -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 . . . . etc

If we extend this to the rationals by throwing in fractions, we get a really dotted line.



Now if we throw in the irrationals with the rationals we get the "real line."



The Complex Plane

It seems natural to put our real numbers in a line, ordered from smaller to larger. The real numbers have the property of being "well ordered". This basically means we can put them in order from least to greatest. If we think of any real number, we can visualize it on the real line and "see" how it relates to the other real numbers.

Now we want to devise a way to visualize the complex numbers. Remember, the real numbers are a subset of the complex numbers. They are the complex numbers with their imaginary part equal to zero.

If we only had the purely imaginary numbers, those of the form, xi, maybe we could just cram them into the real line and call it the reamaginary line. The number $2i$ could be the number that comes after 2, in general, we could put xi after x for every imaginary number. The number $2i$ isn't necessarily bigger or smaller than 2 but we can put things anywhere we want to visualize them easier. This could be a decent way to visualize the purely real and purely imaginary complex numbers together.

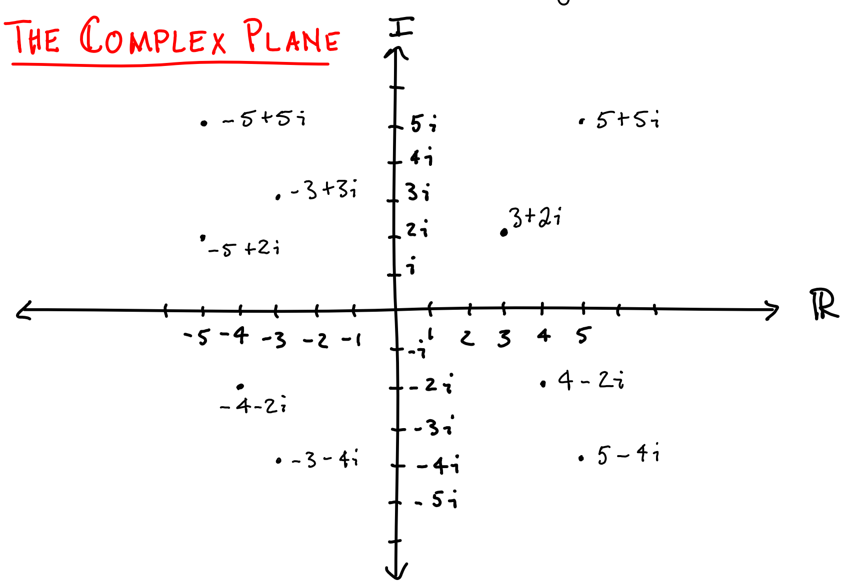
Where could we put the rest of the complex numbers, the ones that aren't purely real or imaginary? Where would we put, say, $3+2i$? What about $2-3i$? These are their own numbers. 3 is a number, $2i$ is a number, and $3+2i$ is also a number. Each one deserves its own special place. Just like 13 is its own number. Sure, it's made of two other numbers, 1 and 3, but 13 gets its own place on the real number line just like 1 and 3.

Is $3+2i < 2-3i$, or is $3+2i > 2-3i$? The answer is neither. The complex numbers, all together, are not well ordered. We can't say which numbers are less than or greater than any other number. We could try cramming $3+2i$ into our "reamaginary line" somewhere, but where would we put it so that we could find it easily? It seems we need to let the line representation go and be a little more creative.

The complex numbers are just a bunch of numbers. If they were in a bag and we dumped them out on our desk, how might we arrange them so that we could see them the easiest?

A good way to do it would be to put the purely real complex numbers in a horizontal line (this is just our real line), and then put the purely imaginary numbers in a vertical line, organized from least to greatest, like so:

THE COMPLEX PLANE



Then we could put $3+2i$ directly above the number 3 and directly to the right of $2i$. We can do this with all of them, <u>as if</u> we were plotting them on an x-y coordinate system. This is what we call the **complex plane**.

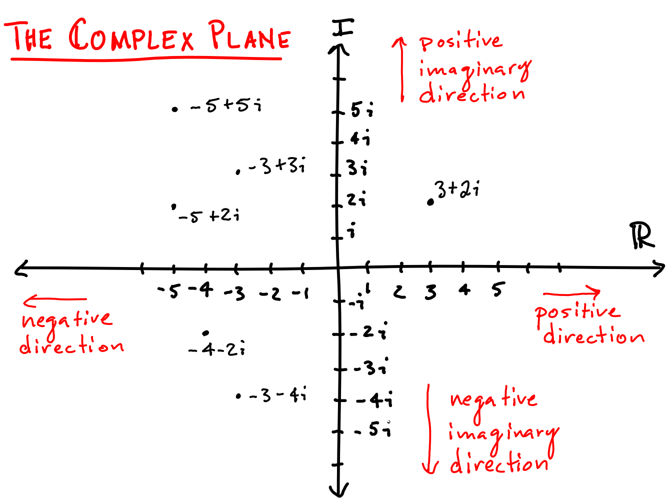
I'm sure you notice it looks a whole lot like the same old x-y coordinate system that we like to draw graphs of functions on. It only looks this way because planes are easy for us to look at, it is a product of our psychology. They are completely different mathematical concepts that happen to share some similarities and are analogous to one another.

What Do Complex Numbers Represent In Reality. How Much Is i?

The purpose of $\sqrt{-1}$ may not be obvious to you at this point. Indeed, it took hundreds of years after its discovery for us to understand what it really is. In many ways its not much different than -1. Negative numbers weren't always around. We don't travel negative distances. No triangle has a side with a negative length. Someone may owe the tax collector a positive $10, but no one actually has negative money.

Negative numbers were once almost as abstract a concept as imaginary numbers. Negative numbers are a powerful mathematical tool once you get used to them and what they mean. They make calculations and mathematical modeling much easier. If the direction you are currently facing is the "positive direction" and distances in that direction are positive distances, then we can consider the direction behind you to be the "negative direction" and if I ask you to take -10 steps, you could take 10 steps backwards. If gaining money is considered to be adding "positive money" to your wallet, then spending money can be considered to be adding "negative money" to your wallet.

Let's take a look at the Complex plane again and consider what i can mean.



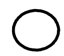
If negative numbers can be considered to just represent the opposite direction then by the looks of the Complex plane, it seems imaginary numbers could almost "naturally" represent the perpendicular direction. For instance, taking $10i$ steps could mean 10 steps to your left and taking $-10i$ steps could be 10 steps to your right.

Complex numbers allow us to represent 2 dimensions without the use of a coordinate system. So, the x-y coordinate system and the complex plane are two different mathematical objects, they can both be used to represent 2 dimensions. We'll see that complex numbers are much, much more than that though.

We should all know what the absolute value of a real number is by now:

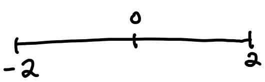
$|2| = 2 \quad |-2| = 2 \quad |-17| = 17$ etc.

Thus, the absolute value of a real number is its distance from zero. This can be thought of as its radius. It may seem odd to say the radius of 7 is 7 but consider this:

A 2 dimensional circle is,... well, a circle 

A 3 dimensional circle is a sphere 

So, what is a 1 dimensional circle, say, of radius 2?

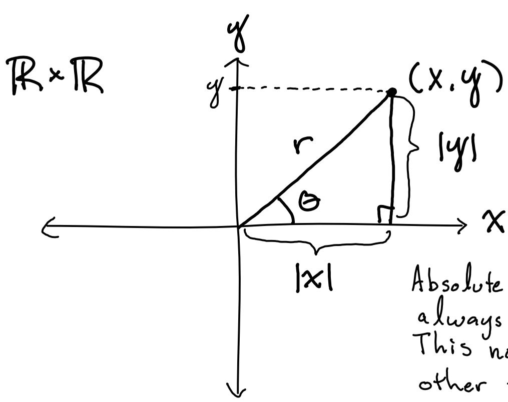
It would be a line segment 

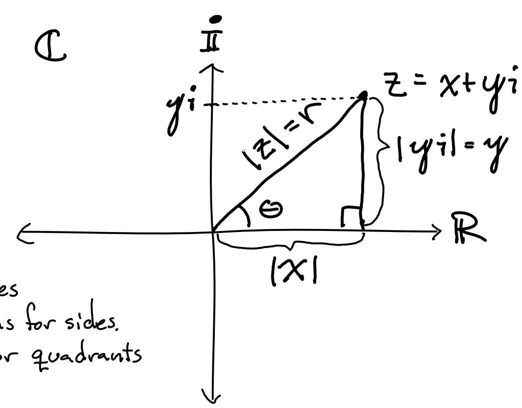
of "diameter" 4.

So, we can think of a radius and the absolute value of a number as the same thing. The "radius" for a complex number is just its absolute value $|z| = r$, also called its modulus or magnitude.

Also, a complex number's angle is sometimes referred to as its argument. Different words are often used for the same concept in different environments. For instance, "length", "area", and "volume" are just different words for "size" in different dimensions. Just as we can use r and θ to specify a point in the x-y plane, we can specify a complex number z in terms of r and θ instead of x and y.

We can use the visual similarities of the Complex plane and the x-y coordinate system. A sketchbook isn't the same thing as a public bathroom stall but people draw on both. Let's draw some triangles.





Absolute values are taken because triangles always have positive real-valued lengths for sides. This notation is obviously necessary for quadrants other than quadrant 1.

So a point in the x-y plane uses an x coordinate and a y coordinate to define that point. These are called Cartesian Coordinates. Alternatively, we could specify r and θ and give pairs of the form (r, θ) instead of (x, y). Coordinates of the form (r, θ) are called Polar Coordinates. From our basic definitions of sine and cosine based on circle geometry, we know:

$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$

$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$

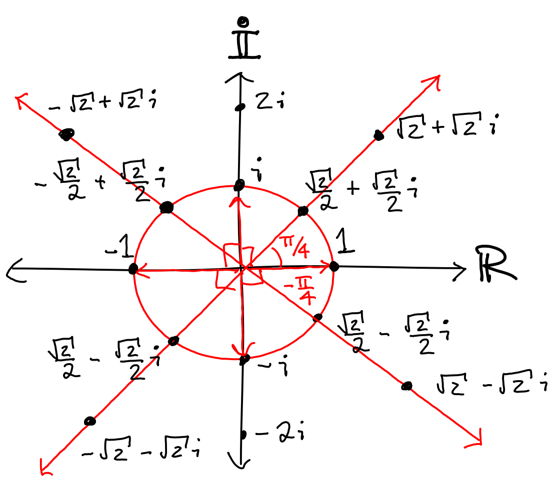
These are our Transformation Equations. They relate x and y to r and θ and therefore provide the means to transfer cartesian coordinates to polar coordinates and vice versa.

So, we can use these equations to represent a complex number z in terms of r and θ instead of x and y. We call this representation its polar form.

$$\underbrace{\boxed{z = x + iy}}_{\text{STANDARD FORM}} = r\cos\theta + ir\sin\theta = \boxed{r[\cos\theta + i\sin\theta] = z} \quad \text{POLAR FORM}$$

Notice we used the term polar form and not polar coordinate. This is because z is a complex number, not a coordinate like (r, θ) which we did call a polar coordinate.

This polar form is a powerful tool, as we'll see shortly, so make sure you get plenty of practice converting complex numbers from standard form to polar form. Going the other way is easy. Let's take a quick look at the complex plane and then do some examples.



- All of the positive real numbers have an angle of zero.
- All of the negative real numbers have an angle of π.
- All of the positive imaginary numbers have an angle of π/2.
- All of the negative imaginary numbers have an angle of 3π/2

Notice $\sqrt{2} + \sqrt{2}i = 2\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right]$ and it has the same angle as $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ with twice the radius.