Part Four: Arithmetic Using Polar Form

If $z_1 = x_1 + y_1 i = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = x_2 + y_2 i = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers then:

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$

There's not much exciting going on there. Notice I put the i after $(y_1 + y_2)$ but I put it before the $(r_1\sin\theta_1 + r_2\sin\theta_2)$. It doesn't matter where you put it, it is just a matter of taste. $i\sin\theta$ looks nicer than $\sin(\theta)i$. If you put the i before the trig function, you don't need to include the parenthesis, but if you put it afterward, it may be ambiguous without them because you can't tell if you mean $\sin(\theta)i$ or $\sin(\theta i)$. On the other hand, $2+3i$ looks better than $2+i3$. When using x's and y's though, $x+iy$ and $x+yi$ both look nice and are commonly used. Remember, $i=\sqrt{-1}$ is just a number so $yi = iy$, so put it where you like it.

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \qquad \text{collect the radii and FOIL}$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2)$$

$$= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1)]$$

$$= \boxed{r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] = z_1\cdot z_2}$$

So, it looks as if you multiply radii and add angles when multiplying complex numbers. Let me say this again except louder:

WHEN MULTIPLYING COMPLEX NUMBERS, MULTIPLY RADII & ADD ANGLES

A simple example: -2 has radius 2 and angle π. 3 has radius 3 and angle zero. $-2\cdot 3 = -6$ which has radius $2\cdot 3 = 6$ and angle $\pi + 0 = \pi$. Thus a negative real number times a positive real number is a negative real number.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \left(\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right)$$

complex conjugates

$$= \left(\frac{r_1}{r_2}\right)\left[\frac{\cos\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2}{\cos^2\theta_2 - i\cos\theta_2\sin\theta_2 + i\cos\theta_2\sin\theta_2 - i^2\sin^2\theta_2}\right]$$

$$= \left(\frac{r_1}{r_2}\right)\left[\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\underbrace{\cos^2\theta_2 + \sin^2\theta_2}_{\to 1}}\right]$$

$$= \boxed{\left(\frac{r_1}{r_2}\right)\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right] = \frac{z_1}{z_2}}$$

So,

WHEN DIVIDING COMPLEX NUMBERS, DIVIDE RADII & SUBTRACT ANGLES

A simple example: -2 has radius 2 and angle π. 3 has radius 3 and angle zero. $-2/3$ has a radius $2/3$ and an angle $\pi - 0 = \pi$. Thus a negative real number divided by a positive real number is a negative real number.

The two "rules" above should seem reasonable. If multiplication implies multiplying radii and adding angles and division is the inverse of multiplying, then it should make sense that dividing implies dividing radii and subtracting angles. Especially since subtraction is the inverse of addition. These "rules" are very useful.

What is $(\sqrt{3}+i)^{12}$? It would really suck to have to multiply this thing by itself twelve times. Luckily, we know **radii multiply and angles add when multiplying complex numbers.** Thus,

$$(\sqrt{3}+i)^{12} = \left[2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\right]^{12} = \left[2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right]^{12} = 2^{12}\left[\cos\left(\frac{12\pi}{6}\right) + i\sin\left(\frac{12\pi}{6}\right)\right]$$

$$= 2^{12}(1 + i\cdot 0) = \boxed{2^{12} = (\sqrt{3}+i)^{12}}$$

We've just multiplied the radius by itself twelve times and added its angle to itself twelve times. We can write this property of powers of complex numbers in a pretty little formula to be found on the following page.

De Moivre's Theorem

$$\boxed{z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

This is just a special case of the fact that radii multiply and angles add when multiplying complex numbers. This is the special case where you're multiplying the same number by itself over and over.

Write $(1+i)^{12}$ in rectangular form.

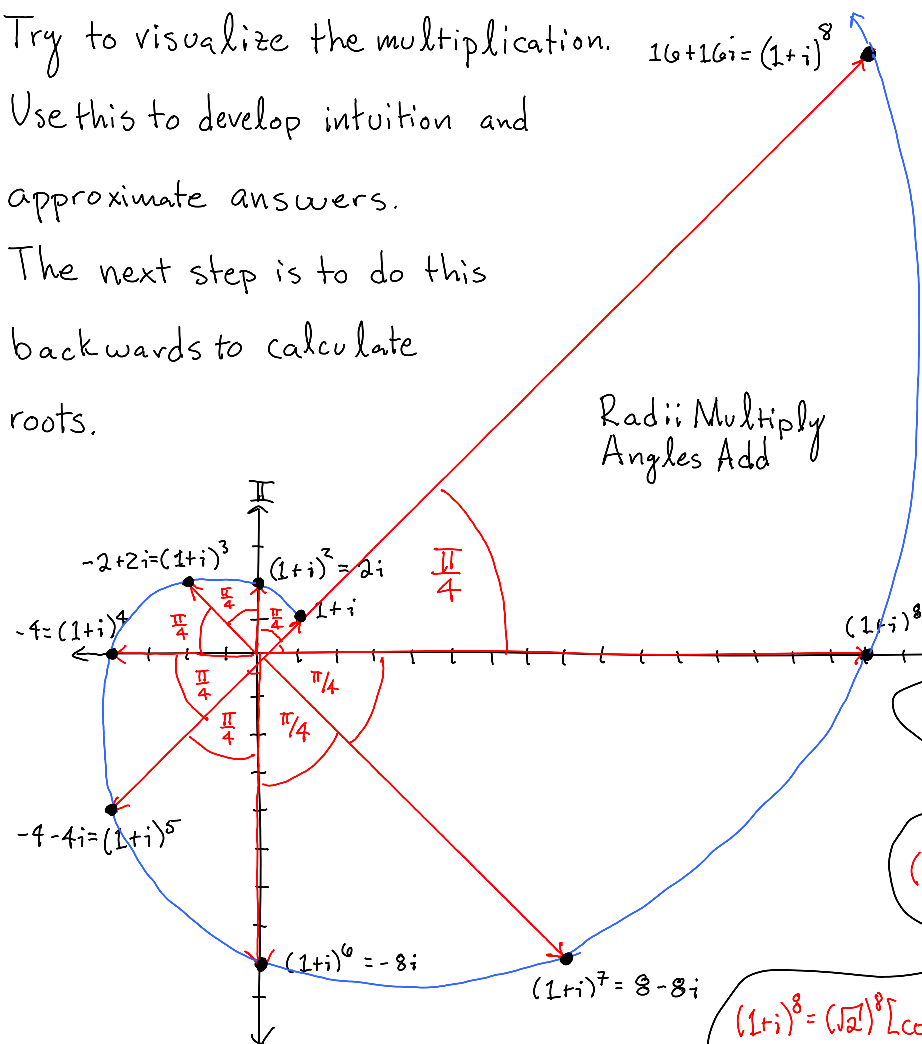
$1+i$ has radius $r = \sqrt{1^2+1^2} = \sqrt{2}$

$\Rightarrow\ 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$ which has an angle $\theta = \pi/4$

$$\Rightarrow (1+i)^{12} = (\sqrt{2})^{12}[\cos(12\ \pi/4) + i\sin(12\ \pi/4)]$$

$$= (2^{1/2})^{12}[\cos(3\pi) + i\sin(3\pi)] = 2^6(-1+0i) = \boxed{-2^6 = (1+i)^{12}}$$

Try to visualize the multiplication. Use this to develop intuition and approximate answers. The next step is to do this backwards to calculate roots.



$(1+i)^2 = (\sqrt{2})^2[\cos(2\ \pi/4) + i\sin(2\ \pi/4)]$

$= 2(0+i) = 2i$

$(1+i)^3 = (\sqrt{2})^3[\cos(3\ \pi/4) + i\sin(3\ \pi/4)]$

$= \sqrt{2}^3(-\sqrt{2}/2 + \sqrt{2}/2\, i) = -2+2i$

$(1+i)^4 = (\sqrt{2})^4[\cos(4\pi/4) + i\sin(4\pi/4)]$

$= 4(-1+0i) = -4$

$(1+i)^5 = (\sqrt{2})^5[\cos(5\pi/4) + i\sin(5\ \pi/4)]$

$= (\sqrt{2})^5(-\sqrt{2}/2 - \sqrt{2}/2\, i) = -4-4i$

$(1+i)^6 = (\sqrt{2})^6[\cos(6\ \pi/4) + i\sin(6\ \pi/4)]$

$= 8(0-i) = -8i$

$(1+i)^7 = (\sqrt{2})^7[\cos(7\ \pi/4) + i\sin(7\ \pi/4)]$

$= (\sqrt{2})^7(\sqrt{2}/2 - \sqrt{2}/2\, i) = 8-8i$

$(1+i)^8 = (\sqrt{2})^8[\cos(8\ \pi/4) + i\sin(8\ \pi/4)] = 16(1+0i) = 16$

Write $(\sqrt{3}-i)^3(2+2i)^6(1-i)^2$ in rectangular form.

$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$ has radius $r=2$ and angle $\theta = -\pi/6$

$2+2i = 2\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$ has radius $r=2\sqrt{2}$ and angle $\theta = \pi/4$

$1-i = \sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$ has radius $r=\sqrt{2}$ and angle $\theta = -\pi/4$

When multiplying complex numbers, radii multiply and angles add, so

$$(\sqrt{3}-i)^3(2+2i)^6(1-i)^2$$

$$= (2)^3(2\sqrt{2})^6(\sqrt{2})^2\left(\cos\left[3\left(-\frac{\pi}{6}\right)+6\left(\frac{\pi}{4}\right)+2\left(-\frac{\pi}{4}\right)\right]+i\sin\left[3\left(-\frac{\pi}{6}\right)+6\left(\frac{\pi}{4}\right)+2\left(-\frac{\pi}{4}\right)\right]\right)$$

$$= 2^{13}\left[\cos(\pi/2)+i\sin(\pi/2)\right] = 2^{13}(0+i) = \boxed{2^{13}i = (\sqrt{3}-i)^3(2+2i)^6(1-i)^2}$$