Part Five : Roots

Taking roots isn't much more complicated than powers. Lets look at some examples:

Find all solutions to $z^4 - 1 = 0$.

$z^4 - 1 = 0 \Rightarrow z^4 = 1$ what can I multiply by itself 4 times to get 1.

Well, 1 has an angle of zero, $0+0+0+0=0$, so 1^4 has angle zero and radius $1 \Rightarrow 1^4 = 1$

That's obvious though. There's another obvious solution, $(-1)^4 = 1$.

-1 has angle π, $\pi+\pi+\pi+\pi = 4\pi$ which is the same as zero and has radius 1.

There are two more. We know $\pi/2 + \pi/2 + \pi/2 + \pi/2 = 4\pi/2 = 2\pi$. So a number with angle $\pi/2$ taken to the 4th power would work if it has a radius of one.

That number is i. $i^4 = i^2 i^2 = (-1)(-1) = 1$

Since $z^4 - 1 = 0$ is a 4th degree polynomial, there are 4 solutions. This fact is brought to you by "The Fundamental Theorem of Algebra". More on that later.

Our fourth solution is $-i$. It has an angle of $3\pi/2$ and radius 1.

$3\pi/2 + 3\pi/2 + 3\pi/2 + 3\pi/2 = 4(3\pi/2) = 6\pi$ which is the same as zero which is the angle of the number 1.

So, $z^4 - 1 = 0$ has the four solutions $\pm 1, \pm i$. Plug any one of these in for z and you'll get zero. Let's take a different approach to finding the solutions. $z^4 - 1$ is the difference of two squares. Let's factor.

$z^4 - 1 = 0 \Leftrightarrow (z^2)^2 - 1^2 = 0 \Leftrightarrow (z^2 - 1)(z^2 + 1) = 0$ but each of these are also differences of two squares. $\Leftrightarrow (z-1)(z+1)(z-i)(z+i) = 0$

Notice these reveal our four solutions. So, if you can factor it, you can find your solutions, or, if you know your solutions, you can write down its factors.

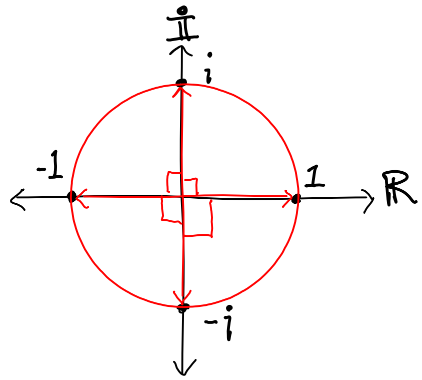
In fact, you can turn many factoring problems into a difference of two squares:

$17 + 3x^2 = (\sqrt{17})^2 - (\sqrt{3}\,x\,i)^2 = (\sqrt{17} + \sqrt{3}\,i)(\sqrt{17} - \sqrt{3}\,i)$

$6x^2 + 2 = (\sqrt{6}\,x)^2 - (\sqrt{2}\,i)^2 = (\sqrt{6}\,x - \sqrt{2}\,i)(\sqrt{6}\,x + \sqrt{2}\,i)$

The i^2 makes the difference!

Let's look at the solutions to $z^4 - 1 = 0$ again. They were $\pm 1, \pm i$.



Notice how the solutions are equally spaced around a circle of radius 1. This is no coincidence. When finding roots the solutions are always equally spaced around a circle.

Let's look at another example.

Find all solutions to $z^5 - 32i = 0$.

$z^5 - 32i = 0 \Leftrightarrow z^5 = 32i \Leftrightarrow z = \sqrt[5]{32i} = \sqrt[5]{32\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]}$ which has 5 solutions, all of which have a radius $\sqrt[5]{32} = 2$. Let's label the angles of the solutions by $\theta_0, \theta_1, \ldots$ or θ_K to represent them all. So, we want angles such that if you add them to themselves 5 times, you get $\pi/2$, or something equivalent. In math, we want θ_K, such that $5\theta_K = \frac{\pi}{2} + 2\pi K$ or $\theta_K = \frac{\pi}{10} + \frac{2\pi}{5}K$; $K = 0, 1, 2, 3,$ and 4

$$z_0 = 2\left[\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right]$$

$$z_1 = 2\left[\cos\left(\frac{\pi}{10} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + \frac{2\pi}{5}\right)\right] = 2\left[\cos\left(\frac{5\pi}{10}\right) + i\sin\left(\frac{5\pi}{10}\right)\right]$$

$$z_2 = 2\left[\cos\left(\frac{\pi}{10} + 2\left(\frac{2\pi}{5}\right)\right) + i\sin\left(\frac{\pi}{10} + 2\left(\frac{2\pi}{5}\right)\right)\right] = 2\left[\cos\left(\frac{9\pi}{10}\right) + i\sin\left(\frac{9\pi}{10}\right)\right]$$

$$z_3 = 2\left[\cos\left(\frac{\pi}{10} + 3\left(\frac{2\pi}{5}\right)\right) + i\sin\left(\frac{\pi}{10} + 3\left(\frac{2\pi}{5}\right)\right)\right] = 2\left[\cos\left(\frac{13\pi}{10}\right) + i\sin\left(\frac{13\pi}{10}\right)\right]$$

$$z_4 = 2\left[\cos\left(\frac{\pi}{10} + 4\left(\frac{2\pi}{5}\right)\right) + i\sin\left(\frac{\pi}{10} + 4\left(\frac{2\pi}{5}\right)\right)\right] = 2\left[\cos\left(\frac{17\pi}{10}\right) + i\sin\left(\frac{17\pi}{10}\right)\right]$$

Verify these are solutions for yourself. (And let me know if I made an error)

Since the angles are shown to come from the fact that we wanted angles θ_K such that $\theta_K + \theta_K + \theta_K + \theta_K + \theta_K = 5\theta_K = \frac{\pi}{2} + 2\pi K$ or $\theta_K = \frac{\pi}{10} + \frac{2\pi}{5}K$ we see that the reason the angles are equally spaced around a circle is just a product of the fact that angles add when multiplying complex numbers.

$\boxed{\text{Roots Are Equally Spaced Around A Circle}}$

So, to find $\sqrt[5]{32i} = \sqrt[5]{\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)}$ our solutions looked like

$$z_K = \sqrt[5]{32}\left[\cos\left(\frac{\pi/2}{5} + \frac{2\pi K}{5}\right) + i\sin\left(\frac{\pi/2}{5} + \frac{2\pi K}{5}\right)\right]$$ where $K = 0, 1, 2, 3, 4 = 5-1$ (compare this to our answers)

What if we wanted $\sqrt[5]{z} = \sqrt[5]{r[\cos\theta + i\sin\theta]}$, the 5-th root of any number z with a radius r and angle θ (We used θ_K for the solutions, don't get them mixed up) Then,

$$\sqrt[5]{z} = \sqrt[5]{r[\cos\theta + i\sin\theta]} = z_K = \sqrt[5]{r}\left[\cos\left(\frac{\theta}{5} + \frac{2\pi K}{5}\right) + i\sin\left(\frac{\theta}{5} + \frac{2\pi K}{5}\right)\right]; \ K = 0, 1, 2, 3, 4 = 5-1$$

The radius of our solutions is $\sqrt[5]{r}$, the 5-th root of the radius of z and the angles of our solutions are $\theta_K = \frac{\theta}{5} + \frac{2\pi K}{5}$.

But what if I want any root for some number z? We'll use the letter "n" to denote the root. Where we had the 5-th root we'll generalize to the n-th root. There will be n solutions with radius $\sqrt[n]{r}$ and the angles will be

$$\underbrace{\theta_K + \theta_K + \theta_K + \cdots \theta_K}_{n\text{-times}} = \underset{\text{solution angle}}{n\theta_K} = \underset{\text{angle of } z}{\theta} + 2\pi K \iff \theta_K = \frac{\theta}{n} + \frac{2\pi K}{n}$$

or anything equivalent to θ

So,

$$\sqrt[n]{z} = \sqrt[n]{r[\cos\theta + i\sin\theta]} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right)\right]; K = 0, 1, \ldots n-1$$

Formula For Finding Roots

The n roots of $\sqrt[n]{z}$ where $z = r[\cos\theta + i\sin\theta]$ are given by

$$\boxed{z_K = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right)\right] \quad K = 0, 1, \ldots, n-1}$$

Let's look at some examples.

Find all solutions to $z^4 - 16i = 0$.

$z^4 - 16i = 0 \Leftrightarrow z^4 = 16i \Leftrightarrow z = \sqrt[4]{16i} = [16(\cos(\pi/2) + i\sin(\pi/2))]^{1/4}$

So our solutions must have a radius of $\sqrt[4]{16} = 2$ and their angles must be such that if we add them to themselves 4 times we get $\frac{\pi}{2}$ or something that points in the same direction as $\pi/2$. If we call the angle for the k-th solution θ_k, then

$4\theta_k = \pi/2 + 2\pi k \Rightarrow \theta_k = \underbrace{\pi/8 + \pi/2\, k}_{} \Rightarrow \theta_0 = \pi/8,\ \theta_1 = \pi/8 + \pi/2,\ \theta_2 = \pi/8 + 2(\pi/2),\ \theta_3 = \pi/8 + 3(\pi/2)$

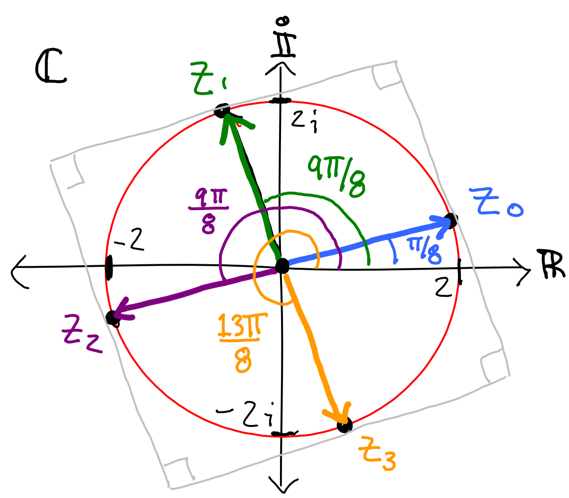
$\boxed{\frac{\theta}{n} + \frac{2\pi k}{n}}$

$z_0 = 2[\cos(\pi/8) + i\sin(\pi/8)]$

$z_1 = 2[\cos(\pi/8 + \frac{\pi}{2}) + i\sin(\pi/8 + \pi/2)] = 2[\cos(5\pi/8) + i\sin(5\pi/8)]$

$z_2 = 2[\cos(\pi/8 + 2\cdot\frac{\pi}{2}) + i\sin(\pi/8 + 2\cdot\frac{\pi}{2})] = 2[\cos(9\pi/8) + i\sin(9\pi/8)]$

$z_3 = 2[\cos(\pi/8 + 3\cdot\frac{\pi}{2}) + i\sin(\pi/8 + 3\cdot\frac{\pi}{2})] = 2[\cos(13\pi/8) + i\sin(13\pi/8)]$



That one was easy enough to do without the formula. Let's do it again using the formula so you see how it works and that it's completely equivalent to what we just did.

Find all solutions to $z^4 - 16i = 0$.

$z^4 - 16i = 0 \Leftrightarrow z = \sqrt[4]{16i}$ so we are looking for the 4, 4-th roots of $16i$.

Our formula says

"The n roots of $\sqrt[n]{z}$ where $z = r[\cos\theta + i\sin\theta]$ are given by

$$z_K = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}K\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi}{n}K\right)\right] \quad K = 0, 1, \dots, n-1 \text{"}$$

Since were looking for $\sqrt[4]{16i}$, our n is 4.

$16i$ has angle $\theta = \pi/2$ and radius 16

So, after plugging in $n=4$, the formula gives

$$z_K = \sqrt[4]{r}\left[\cos\left(\frac{\theta}{4} + \frac{2\pi}{4}K\right) + i\sin\left(\frac{\theta}{4} + \frac{2\pi}{4}K\right)\right] \quad K = 0, 1, \dots, 4-1=3$$

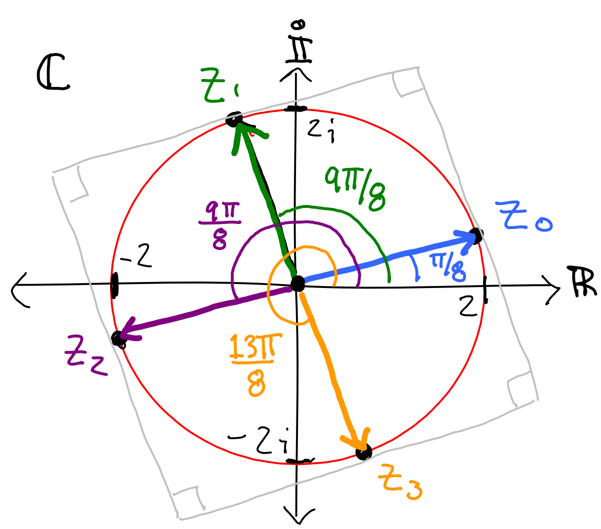
plugging in $r = 16$ and $K = 0, 1, 2,$ and 3 gives our 4 solutions

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 0\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 0\right)\right] = 2\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right]$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 1\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 1\right)\right] = 2\left[\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right]$$

$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 2\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 2\right)\right] = 2\left[\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right]$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 3\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2\pi}{4}\cdot 3\right)\right] = 2\left[\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)\right]$$



NOTE:

Notation can be very confusing. When we write $\sqrt{4}$, we mean 2, not ± 2, as said before. You get both when solving the equation: $x^2 = 4 \Leftrightarrow x = \pm\sqrt{4} = \pm 2$

When you're only dealing with real numbers this works fine. But if we're in a complex environment and we see $\sqrt{3-2i}$, what does this mean?

Is it the "positive" choice as before? "Positive" doesn't have much meaning in the complex world, so we must mean both roots. Likewise $\sqrt[4]{16} = 2$ if we are only considering real numbers, like when we're finding a radius, which is always a positive real number. But $\sqrt[4]{3-2i}$ refers to all 4 roots. $\sqrt[4]{16}$ can mean all 4 roots too if we're in a complex environment and not refering to a radius. I prefer to write "the solutions to $z^4 = 16$" when I want all four roots and $\sqrt[4]{16}$ if I just want 2 for an answer when dealing with roots of real numbers in complex environments as to avoid ambiguity. Mathematical notation can be a lot like English sometimes. There's slang, multiple words with the same meaning, words spelled the same but pronounced differently and may or may not have different meanings. It's all understood via the context of the situation. Remember, Math is a language and must be treated as one. Memorizing a Swahili dictionary probably won't help you follow a conversation in Swahili much. You have to pratice and join in on the conversation even if you're not sure of the right words. You point and grunt and make funny gestures along with your broken Swahili to get the point across. The more you work at it, the less pointing and grunting you have to do. Eventually you may become an elegant speaker of Swahili if you practice enough.

Of course you can prentend to practice but not actually learn much. You can make really refined points and grunts but you're still pointing and grunting. You may have met someone from some far-off country that has lived here for 40 years and can just barely formulate a simple sentence in English They were immersed in the language but never really tried to learn it. They just made their way by on an "as needed" basis. Likewise, you can make your way by through years of math classes and never really know what's going on. Just smiling and knodding can get you through the situation repeatedly but its never a very comfortable situation.

Join the conversation and "learn" to have fun with it.