# Sum And Difference Formulas

$$\boxed{\begin{aligned}\cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\end{aligned}}$$

## Notation

The "$\pm$" and "$\mp$" symbols provide a way to conveniently sum up 2 equations in one. For instance,

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \iff \begin{cases}\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{cases}$$

Notice the plus and minus are reversed here. The top symbols go together to form one equation and the bottom symbols go together to form the other.

You **MUST** remember these. Put them right next to the Pythagorean Theorem in your head. With their powers combined, you'll be able to derive all of the identities you need to know in 1 or 2 steps (3 at the most).

## How To Remember Them

Here's how you can conjure them from your head when you need them:

Cosine is an even function, so $\cos(\alpha\pm\beta)$ must equal something even. Apply the same idea to sine, which is an odd function.

even = even

$$\underset{\text{even}}{\cos(\alpha\pm\beta)} = \underbrace{\underbrace{\underset{\text{even}}{\cos\alpha} \cdot \underset{\text{even}}{\cos\beta}}_{\text{even}} \mp \underbrace{\underset{\text{odd}}{\sin\alpha} \cdot \underset{\text{odd}}{\sin\beta}}_{\text{even}}}_{\text{even}}$$

even → even

odd = odd

$$\underset{\text{odd}}{\sin(\alpha\pm\beta)} = \underbrace{\underbrace{\underset{\text{odd}}{\sin\alpha} \cdot \underset{\text{even}}{\cos\beta}}_{\text{odd}} \pm \underbrace{\underset{\text{even}}{\cos\alpha} \cdot \underset{\text{odd}}{\sin\beta}}_{\text{odd}}}_{\text{odd}}$$

odd → odd

Want $\cos(\alpha\pm\beta)$? Then $\cos\alpha$ comes first.

$$\cos(\alpha\pm\beta) = \underbrace{\cos\alpha\cos\beta} \mp \underbrace{\sin\alpha\sin\beta}$$

Change the sign

Match the trig.

Want $\sin(\alpha\pm\beta)$? Then $\sin\alpha$ comes first.

$$\sin(\alpha\pm\beta) = \underbrace{\sin\alpha\cos\beta} \pm \underbrace{\cos\alpha\sin\beta}$$

Keep the sign.

Mix the trig.

## How To Remember Them. Part 2. What If I Can't Remember?

Well, you'd better figure out how to remember it, or my friend Moe might poke you in the eye.

| | |
|---|---|
| $\cos(\alpha \pm \beta)$ = even | $\sin(\alpha \pm \beta)$ = odd |
| Start with $\cos\alpha$ | Start with $\sin\alpha$ |
| Match the trig | Keep the sign |
| Change the sign | Mix the trig |

The odd and even observation will be true for any identity and this is not something to remember but is something that should make intuitive sense. So, let's say you understand that both sides of the identity must both be even or odd and you know that $\sin\alpha$, $\sin\beta$, $\cos\alpha$, and $\cos\beta$ show up but you don't know where. The symmetry (odd and evenness) should tell you to match the trig for cosine and mix it for sine, but you don't remember what's negative and what's positive. **Write down the closest thing you can think of and plug in some values.** Are both sides equal? If so, try plugging in a few other values just to be safe. If not, try to adjust your "almost identity" to accomodate for the error. You might be able to figure it out and turn your "almost identity" into a real one.

You only really need to know the sum formulas.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

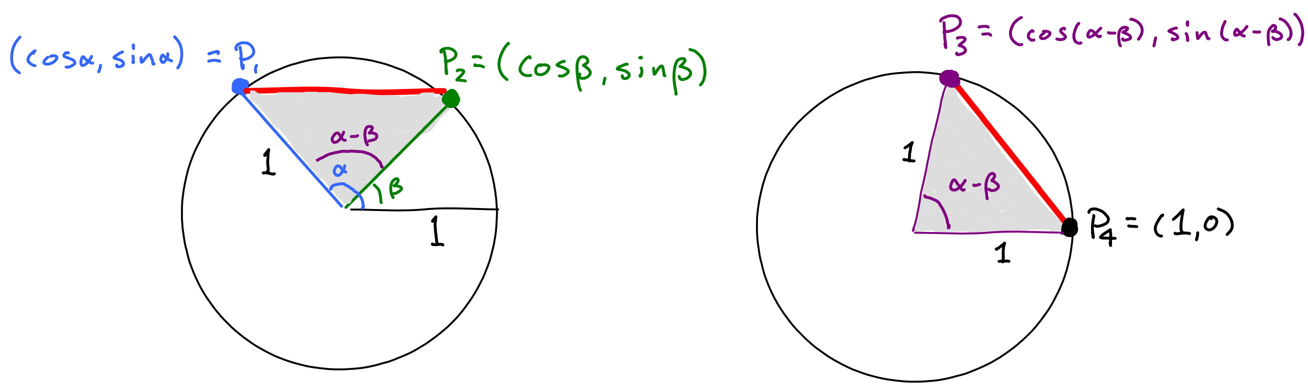
The difference formulas are imbedded in them because nothing says that $\beta$ can't be negative. You can simply add a negative angle. $\sin(\alpha-\beta) = \sin[\alpha+(-\beta)]$

How do we know these formulas are true? How did someone figure them out? I'll show you one way to derive them.

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

## Proof

After wondering how to relate $\cos(\alpha \pm \beta)$ with $\cos\alpha$, $\sin\alpha$, $\cos\beta$, or $\sin\beta$ for a while, you may find that drawing a picture could help. After a couple of quick drawings I found that it could be easier to find the identity for the difference $\cos(\alpha-\beta)$ first after noticing the two triangles below are the same triangle. The red sides of the two triangles below are equal since they are really the same triangle. This provides a link between $\cos(\alpha-\beta)$ and $\cos\alpha, \cos\beta, \sin\alpha$, and $\sin\beta$. We can use the distance formula to find the lengths of the red line in both triangles and then equate them.




**Distance Formula:** If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ then the distance between point $P$ and point $Q$ is given by $\boxed{D(P,Q) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}}$

Notice that if one of your points is the origin $(0,0)$, the Distance formula reduces to the Pythagorean Theorem.

Using the Distance Formula and our four points

$P_1 = (\cos\alpha, \sin\alpha)$, $P_2 = (\cos\beta, \sin\beta)$, $P_3 = (\cos(\alpha-\beta), \sin(\alpha-\beta))$, and $P_4 = (1,0)$ we know:

$$D(P_1, P_2) = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

$$D(P_3, P_4) = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2}$$

We also know $D(P_1, P_2) = D(P_3, P_4)$ because these are the lengths of the red sides of the triangles above. So, we have

$$D(P_1, P_2) = D(P_3, P_4)$$

$$\Leftrightarrow \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2}$$

(square both sides)

$$\Leftrightarrow (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = (\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2$$

(Multiply the squares out)

$$\Leftrightarrow \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta = \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)$$

$$1 \qquad\qquad 1 \qquad\qquad 1$$

$$\Leftrightarrow \cancel{2} - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = \cancel{2} - 2\cos(\alpha-\beta)$$

(subtract 2 from both sides and multiply by $-1/2$)

$$\Leftrightarrow \boxed{\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)}$$

That was all of the real work. With this, the rest is easy (easier).

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Now, use your cofunction identities, $\sin\theta = \cos(\pi/2 - \theta)$ and $\cos\theta = \sin(\pi/2 - \theta)$.

Since $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, we have

$$\sin(\alpha+\beta) = \cos[\pi/2 - (\alpha+\beta)] = \cos(\pi/2 - \alpha - \beta) = \cos[(\pi/2 - \alpha) - \beta]$$

$$= \cos(\pi/2 - \alpha)\cos\beta + \sin(\pi/2 - \alpha)\sin\beta$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

So, $\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

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Now, use your odd and even identities, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Since $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, we have

$$\cos(\alpha+\beta) = \cos(\alpha - (-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

So, $\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$ and since $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, we have

$$\sin(\alpha-\beta) = \sin[\alpha+(-\beta)] = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

So, $\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$

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## EXAMPLE

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$\pi/12 = 15°$ is not an angle labeled on our unit circle but we can find exact values for $\sin(15°)$ and $\cos(15°)$ using our new identities.

$$\sin(15°) = \sin(45°-30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4} = \sin(15°) = \sin(\pi/12)}$$

$$\cos(15°) = \cos(45°-30°) = \cos(45°)\cos(30°) + \sin(45°)\sin(30°)$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4} = \cos(15°) = \cos(\pi/12)}$$

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So the basic trick is to write an angle that is unlabeled on the unit circle as a sum or difference of two angles that you do know.

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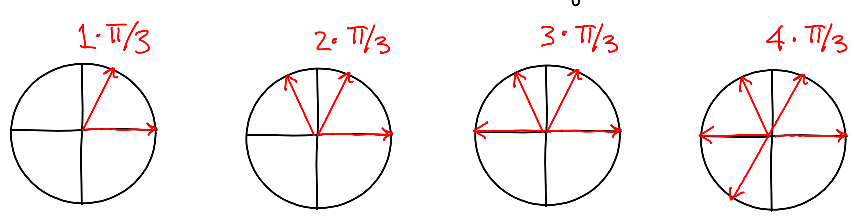
Normally I like to work with radians because it takes less effort to write $\pi$ than it does to write 180°. Also, I find it easier to navigate the unit circle using radians.

$\frac{4\pi}{3}$ is four $\frac{\pi}{3}$'s




It's easy for me to count multiples of $\pi/3$, $\pi/4$, or $\pi/6$ and visualize it in my head in order to relate the given angle to what I know about the angles in Quadrant 1. However, when dealing with sums and differences, I find it easier to work in degrees. $15° = 45° - 30° = 60° - 45°$ versus $\pi/12 = \pi/3 - \pi/4 = \pi/4 - \pi/6$

# The Unit Circle: Expanded

With our new tool we can really expand what we know about the unit circle. We've already expanded our unit circle to include labels for $15° = \pi/12$. We can do the same for many others using the same technique.

$$\sin(75°) = \sin(45° + 30°) = \sin(45°)\cos(30°) + \cos(45°)\sin(30°)$$

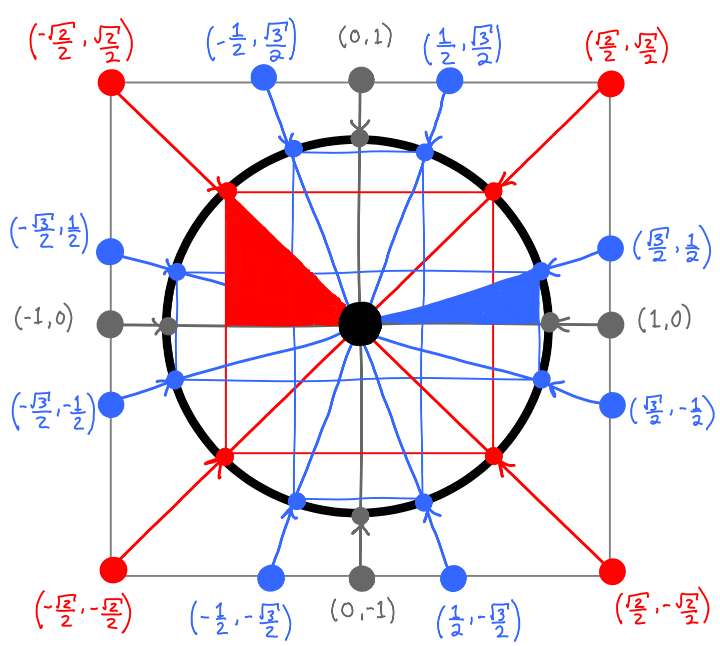
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4} = \sin(75°) = \sin(5\pi/12)}$$

But of course we should have realized this without doing the calculation since we already knew $\cos(15°) = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\cos(15°) = \sin(75°)$ through our cofunction identities. $\sin(75°) = \sin(90° - 75°) = \cos(15°)$

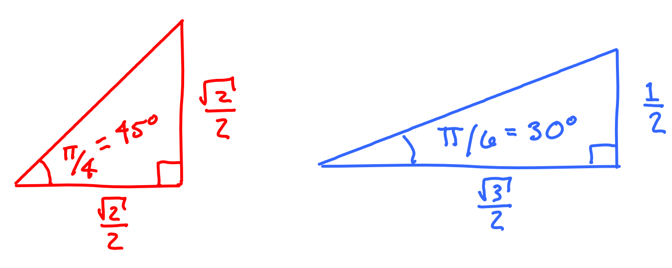
It seems our cofunction identities are just special cases of our difference formulas, the special case where the first angle is $90°$.

Before doing more calculations, let's take a visual look at what we've done.

## The Basic Unit Circle



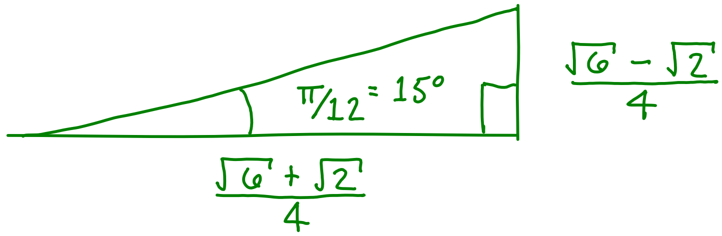

This is the old unit circle. We get all of these angles just by knowing the sides of the two following triangles:




Each angle in our unit circle can be created by placing one of these two triangles in the appropriate place and putting minus signs where necessary. Also, you can produce most of the coordinates using only $\frac{1}{2} = \frac{\sqrt{1}}{2}$, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{2}$.

# The New and Improved Unit Circle (150% More Angles)

We can greatly extend the old unit circle by tossing in our knowledge of a new triangle.

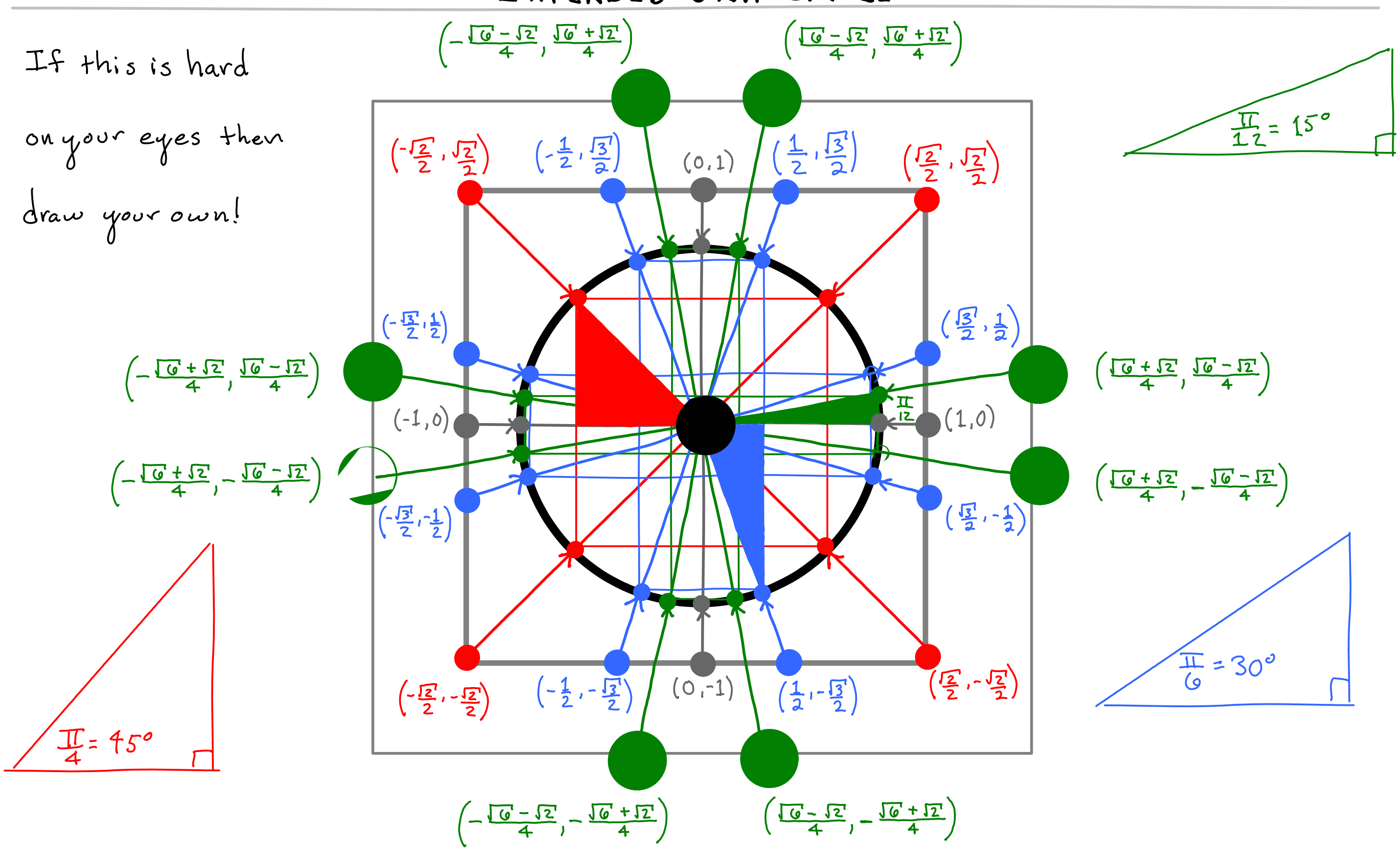



Now, just throw in the numbers $\frac{\sqrt{6}\pm\sqrt{2}}{4}$ with $\frac{1}{2}=\frac{\sqrt{1}}{2}$, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{2}$.

Notice the longer side has the plus in the middle and the shorter side has the minus. I usually remember that the numbers $\frac{\sqrt{6}\pm\sqrt{2}}{4}$ have a 2, 4, and a 6 in them and the 2 and the 6 are square rooted. The 4 isn't square rooted because $\sqrt{4}=2$. I also remember that the 4 goes on the bottom because books don't like to put roots on the bottom of the fraction.

## Extended Unit Circle

If this is hard on your eyes then draw your own!



# Sum And Difference Formulas For Tangent

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

We could leave it like this, but since it is a formula for tangent, maybe we should convert the assortment of sines and cosines to an assortment of tangents. There are any number of ways to do this but we only need one. A "smart one" that is. We want to multiply or divide the top and bottom of this fraction by something that will make the cosines and sines go away and leave us with tangents. The key here is $\cos\alpha\cos\beta$. Divide the top and bottom of this beast by it and you'll see why.

$$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \left[\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\right]\left[\frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}\right]$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \boxed{\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha+\beta)}$$

Take a look at what we started with, what we ended with and how we figured out how to get there. How we got there is not even close to being as important as how we figured out how to get there. Anyone can follow the steps. You want to learn to lead. It's like playing one of those electronic musical keyboards with the blinking lights that tell you which key to press next. We can teach a monkey to follow the lights and play the William Tell Overture for a banana and a wooden stick specially shaped so he can scratch the middle of his back with it. Of course we haven't taught him to play the piano, we taught him how to follow blinking lights.

## An Awkward Example

Show that $\tan(\pi/2 + \theta) = -\cot\theta$ is an identity.

We can try our new formula:

$$\tan(\pi/2 + \theta) = \frac{\tan(\pi/2) + \tan\theta}{1 - \tan(\pi/2)\tan\theta} = \text{undefined}$$

Houston, we have a problem. Tangent is undefined at $\pi/2$. Fortunately neither sine nor cosine has this problem. We introduced this issue when we converted our formula from sines and cosines to tangents by dividing everything by $\cos\alpha\cos\beta$. We just need to take a step backward and use the uglier but more encompassing version of our formula:

$$\tan(\pi/2 + \theta) = \frac{\sin(\pi/2 + \theta)}{\cos(\pi/2 + \theta)} = \frac{\sin(\pi/2)\cos\theta + \cos(\pi/2)\sin\theta}{\cos(\pi/2)\cos\theta - \sin(\pi/2)\sin\theta}$$

$$= \frac{1\cdot\cos\theta + 0\cdot\sin\theta}{0\cdot\cos\theta - 1\cdot\sin\theta} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

Everything worked fine this time. Notice that $\cos(\pi/2) = 0$, so dividing by $\cos(\pi/2)$ is the same as dividing by zero which is a big No No. This is why our previous attempt failed.

I thought identities held true no matter what numbers we plugged in?

$$\sin(x) = \frac{\sin(x)\cos(x)}{\cos(x)} = \tan(x)\cos(x)$$

This is an identity but $\sin(x)$ will take any value for $x$ and tangent is undefined at $x = \pi/2$. So, identities are valid for any value of $x$ such that both sides of the equation are defined. Consider $1 = x/x$.

Usually the sum formula for tangent works fine but if you run into problems with it being undefined, you'll need to revert to your cosines and sines.