How To Derive The Quadratic Formula

If you run into a polynomial of second degree that you can't factor while trying to solve for the unknown variable, you've undoubtedly used the quadratic formula.

The Quadratic Formula

$$\boxed{ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

The a, b, and c represent the coefficients and x is your unknown.

Example: Solve for x. $x^2+3x+2=0$

$$a=1,\ b=3,\ c=2 \Rightarrow x=\frac{-3\pm\sqrt{3^2-4(1)(2)}}{2(1)}=\frac{-3\pm 1}{2}$$

$\Rightarrow$ The two solutions to our 2nd degree polynomial (also known as a quadratic) are $x=\frac{-3+1}{2}=-1$ and $x=\frac{-3-1}{2}=-2$

Plug them in the equation to verify:

$$(-1)^2+3(-1)+2=1-3+2=0\ \checkmark$$

$$(-2)^2+3(-2)+2=4-6+2=0\ \checkmark$$

The quadratic formula simply rearranges the equation and solves for x. You can use it for any quadratic equation even if you could factor instead.

Example: Solve for y. $y^2-3y-4=0$

$$y^2-3y-4=0 \Leftrightarrow (y+1)(y-4)=0 \Leftrightarrow y=-1 \text{ or } y=4$$

Or with the quadratic formula: $y=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-4)}}{2(1)}=\frac{3\pm 5}{2}=4,-1$

This formula isn't magic and you shouldn't need a mnemonic or have to sing a lousy song in order to remember it. You should be able to derive it with ease. I'll show you how, but you need to focus on "why" so you can apply the perspective needed for this derivation to other problems.

The key trick is called "completing the square." This involves adding the right amount to both sides of the equation so that you get something squared on the side of the equation with the unknown. Here's an example.

Example: Solve for x. $x^2 + 3x + 2 = 0$

$$x^2 + 3x + 2 = 0 \iff x^2 + 3x + \frac{9}{4} = \frac{9}{4} - 2 \iff \left(x + \frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\iff x + \frac{3}{2} = \pm\sqrt{\frac{1}{4}} \iff x = -\frac{3}{2} \pm \frac{1}{2} = \frac{-3 \pm 1}{2} = -1, -2$$

This was the same example as was done earlier using the quadratic formula. In fact, we did the same thing that the quadratic formula does - completes the square. How did I know to add 9/4 to each side of the equation?

Compare $(p+q)^2 = p^2 + 2pq + q^2$ to $x^2 + 3x + 2$

The x^2 matches the p^2 so our $p = x$. Now focus on the middle term:

$2pq = 2xq$ compared to our middle term $3x$.

We need to find q and make sure we have q^2 to go with our $p^2 + 2pq$.

$$2pq = 2xq = 3x \iff q = \frac{3x}{2x} = \frac{3}{2} \iff q^2 = \frac{9}{4}$$

This is why we add 9/4 to each side:

$$x^2 + 3x + 9/4 + 2 = (x + 3/2)^2 + 2$$

Never mind the extra 2, just throw it to the other side of the equation like we did in the example.

We focus on the terms with the x's because that is what we are trying to isolate. We can always add and subtract any constant we want like 9/4. Make up a few problems of your own and practice completing the square so you get the idea. You can complete the square no matter what your coefficients are. Let's call them a, b, and c.

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Get rid of that pesky number in front of the x^2.

$$\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Isolate the unknown to one side of the equation.

Now find the amount to add to each side to complete the square. Remember, the a, b, and c are just numbers. They were the original coefficients.

$$(p+q)^2 = p^2 + 2pq + q^2 \quad \text{compared to} \quad x^2 + \frac{b}{a}x$$

$$p = x \qquad 2pq = 2xq = \frac{b}{a}x \Leftrightarrow q = \frac{bx}{2ax} = \frac{b}{2a} \Leftrightarrow q^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Our q is simply half of the middle coefficient

Now, complete the square.

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \Leftrightarrow x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

and then follow through.

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Leftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There it is: $ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Derive it on your own every few days until it is obvious. You should eventually see through the steps, so don't memorize the steps. That's worse than memorizing the formula. Use your own symbols and style and develop some intuition. Write your own tutorial so you never forget.

A Closer Look

There are two solutions to the equation $ax^2 + bx + c = 0$. This follows from the Fundamental Theorem of Algebra. It is also apparent from the presence of the "$\pm$" in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The Discriminant: $\sqrt{b^2 - 4ac}$

- If the stuff inside the square root is negative, then both solutions will have imaginary parts. ie. $b^2 < 4ac \Rightarrow x \in \mathbb{C}$

Example: Solve for x: $x^2 + 2x + 3 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} = -1 \pm \frac{\sqrt{-8}}{2} = -1 \pm \sqrt{2}\,i$$

- If the stuff inside the square root is positive, then both solutions will be purely real. ie. $b^2 > 4ac \Rightarrow x \in \mathbb{R}$

Example: Solve for x: $x^2 - 2x - 2 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = 1 \pm \frac{\sqrt{12}}{2} = 1 \pm \sqrt{3}$$

- If the stuff inside the square root is zero, then both solutions will be the same purely real number. When you get the same solution twice, we say the solution has multiplicity of two. ie. $b^2 = 4ac \Rightarrow x_1 = x_2$ where $x_1, x_2 \in \mathbb{R}$

Example: Solve for x: $x^2 + 2x + 1 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} = -1 \pm 0 = -1$$

Notice: $x^2 + 2x + 1 = 0 \Leftrightarrow (x+1)(x+1) = 0 \Leftrightarrow x = -1, x = -1$

Complex Coefficients

The Quadratic Formula still works even if the coefficients a, b, or c have complex values. (Remember $i \equiv \sqrt{-1} \Rightarrow i^2 = -1$)

Example: Solve for x: $x^2 + 2ix + 1 + i = 0$

Here $a = 1$, $b = 2i$, and $c = 1+i$

$$x = \frac{-2i \pm \sqrt{(2i)^2 - 4(1)(1+i)}}{2(1)} = \frac{-2i \pm \sqrt{-8-4i}}{2} = -i \pm \sqrt{-2-i}$$

Check: $\left[-i \pm \sqrt{-2-i}\right]^2 + 2i\left[-i \pm \sqrt{-2-i}\right] + 1 + i$

$$= -3 - i \mp 2i\sqrt{-2-i} + 2 \pm 2i\sqrt{-2-i} + 1 + i = 0 \checkmark$$

Example: Solve for x: $x^2 + (1+i)x - (2+i) = 0$

Here $a = 1$, $b = 1+i$, and $c = -(2+i)$

$$x = \frac{-(1+i) \pm \sqrt{(1+i)^2 + 4(1)(2+i)}}{2(1)} = \frac{-(1+i) \pm \sqrt{8+6i}}{2}$$

Check: $\left[\frac{-(1+i) \pm \sqrt{8+6i}}{2}\right]^2 + (1+i)\left[\frac{-(1+i) \pm \sqrt{8+6i}}{2}\right] - (2+i)$

$$= \frac{8 + 8i \mp (1+i)\sqrt{8+6i}}{4} + \frac{-2i \pm (1+i)\sqrt{8+6i}}{2} - (2+i)$$

$$= \frac{8 + 8i \mp 2(1+i)\sqrt{8+6i}}{4} - \frac{4i \mp 2(1+i)\sqrt{8+6i}}{4} - \frac{8+4i}{4}$$

$$= \frac{8 + 8i \mp 2(1+i)\sqrt{8+6i}}{4} - \frac{8 + 8i \mp 2(1+i)\sqrt{8+6i}}{4} = 0 \checkmark$$